PARAMETRIC CONDITIONS FOR SIMULTANEOUS QUADRATIC STABILIZATION OF LINEAR SYSTEMS

A Thesis Submitted
in Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

by
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to the
DEPARTMENT OF ELECTRICAL ENGINEERING
INDIAN INSTITUTE OF TECHNOLOGY, KANPUR
May, 1996

CERTIFICATE

Certified that the work contained in the thesis entitled "*Parametric Conditions for Simultaneous Quadratic Stabilization of Linear Systems*", by "*Indra Narayan Kar*", has been carried out under my supervision and that this work has not been submitted elsewhere for a degree

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Synopsis

An important problem in control engineering is to design a single feedback control law which will stabilize finite number of plants. This is known as the simultaneous stabilization (SS) problem. This problem is motivated by several practical considerations, for instance, the dynamics of an aircraft varies greatly with its altitude and speed and requires stabilization at all flight conditions. An industrial plant is required to remain stable in different modes due to possible failures of components such as sensors and actuators. Often a linearized model of a non-linear system operating at different operating points has time varying parameters. In such situations, it is important to design a feedback controller which will stabilize the system in spite of such changes. This thesis is aimed at studying the simultaneous quadratic stabilization problem.

Consider a finite collection of Linear Time Invariant (LTI) systems denoted by $\{A_i, B_i\}$, $i = 1, 2, \ldots, l$ and described by the state equations

$$\dot{x} = A_i x + B_i u$$

where $x \in R^n$ is the state vector and $u \in R^m$ denotes control input. The basic problem considered in this thesis is to find a single state feedback control law $u = -Kx$ such that there is a quadratic Lyapunov function $V(x) = x^* P x$ (P being a positive definite matrix) which has negative time derivative along the solution of each of the closed loop systems. This problem is referred to as the Simultaneous Quadratic (SQ) stabilization problem. More specifically, SQ stabilization problem involves the design of a single control law $u = -Kx$ and a positive definite matrix P such that the derivative of a common quadratic Lyapunov function $V(x) = x^* P x$ will be negative definite for the solutions of each of the closed loop system. For a general finite set of systems $\{A_i, B_i\}$, the problem of showing the existence of K and P is quite difficult and open. In the present thesis, we attempt to identify different structures of state space matrices $\{A_i\}$ and $\{B_i\}$ for which

SQ stabilization problem is solvable. The thesis is outlined as follows.

Chapter 1 discusses the importance of simultaneous stabilization problem in control theory with a brief literature review of the methods of controller design for uncertain systems.

In chapter 2, definitions and preliminary results for SQ stabilization are introduced. It is then established that if a finite set of systems $\{A_i, B\}$, with fixed input matrix B, are simultaneously transformable into **Controllable Companion** (CC) form or more generally into the **Hessenberg** form (called as Hessenberg family) then they are SQ stabilizable. Algorithms are also proposed to achieve stabilization using a single linear static state feedback controller It turns out that the family of systems, which are simultaneously transformable into CC form, satisfy the matching condition. However, the Hessenberg family is shown to be larger than the family satisfying the matching condition. At the end of this chapter, conditions for SQ stabilization of a set of systems $\{A_i, B_i\}$, with uncertain input matrix, are discussed.

Chapter 3 reports three new families of systems which are SQ stabilizable. These families of systems are called (1) **partially commutative** (2) **partially normal** and (3) **a more general** family Each of these families is defined by imposing certain assumptions on the state space matrices $\{A_i\}$ and $\{B_i\}$ These families of systems are shown to be different from the matched uncertain systems. For each case the existence of a simultaneous quadratic stabilizing controller is ensured.

For state feedback controller, it is necessary to measure all the state variable for feedback. However in practice, the design of output feedback controller is a realistic approach. Chapter 4 is devoted to the design of static output feedback controller for the simultaneous quadratic stabilization problem. For this purpose, we consider a finite collection of minimum phase systems $\{A_i, B_i, C\}$ described by

$$\dot{x} = A_i x + B_i u, \ y = Cx$$

where y is the output vector. These systems are, in addition, assumed to be square systems (equal number of inputs and outputs).

Two new classes of systems are then defined which are SQ stabilizable by output feedback controller. An algorithm for computing a single static output feedback stabilizing controller is also presented.

Chapter 5 is concerned with the design of a simultaneous stabilizing controller for

time delay systems and stabilization of time varying systems. Effectively, the results of chapter 2 and 3 are extended for special classes of time delay systems and the existence of a single stabilizing controller is proved. More specifically, different class of time delay systems are shown to be SQ stabilizable by a single memoryless state feedback controller. The stabilization of time delay systems is addressed here using a quadratic Lyapunov functional. This chapter concludes with the design of a single stabilizing controller for special classes of time varying systems

Finally, in chapter 6. we take two practical examples for which the solution of SQ stabilization problem is applied. These examples are (1) stabilizing controller design for an uncertain aircraft model (2) Tracking controller design for a robot manipulator

In this thesis, parametric conditions (structures of system matrix A_i and B_i) are derived for which SQ stabilization problem is solvable Chapter 7 summarizes all such parametric conditions and the overall conclusions of the thesis.

Acknowledgements

I wish to express my deep sense of gratitude to my supervisor Dr. V. R. Sule for his guidance, advice and encouragement. It was a rewarding experience for me to interact with him and gain insight into the topics of research. The various values that I tried to learn from him shall remain a source of inspiration for me forever. I owe it to my teachers particularly professors A. Ghosh, S S Prabhu, V.P. Sinha, R.N. Biswas, P.R.K. Rao, B. Sarkar, P. Sircar, K.E. Hole, A.K Raina for the deep insights given through the various courses they taught.

During the course of this thesis, a significant fraction of time has been spent in the company of my various friends. We have discussed a lot of events, issues, ideas and people threadbare. Together we have utilized and wasted quite a lot of time. I have greatly benefited from the cooperation and help extended by them. In this regard I am thankful to Joy, Pavitra, Arun, Sandeep, Nilu, Susa, Anirban, Biswajit, Joydeep, Deepak, Premananda, Soumya and Shymal. I am also specially thankful to Anirban and Joy for their willingness to discuss various technical ideas with me.

I have benefited greatly from the interaction with many research scholars. For this, I would like to thank Shafi, Deshpande, Shivnarayan, Chaitanya, Biswarup, Kishore, Puranjoy, Sudipta, Balbindar, Babu, GK, Dandapat, Ramesh, Vinod, Jitendra, Tsr, Apu, Harish, Rahul.

I am gratefull to Metia da and Pal da for their homely affection and enquires about me.

It was impossible to venture into the Ph.D programme without the support and encouragement of my family members. They have willingly taken up difficult tasks to make me feel comfortable. I thank them for patiently bearing my repeated assurance that thesis will be completed in notime.

Indra Narayan Kar

DEDICATED to

my Parents

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List of Symbols and Abbreviations

$\mathbf{0}$ · The zero matrix or vector of appropriate dimension

$A < \mathbf{0}$: A is a negative definite matrix.

$A > \mathbf{0}$: A is a positive definite matrix.

$A > (<) B$: $A - B$ is a positive (negative) definite matrix.

I_m : $(m \times m)$ identity matrix.

W : A positive definite matrix.

$\lambda_M[A]$: Maximum eigenvalue of the matrix A.

$\lambda_m[A]$: Minimum eigenvalue of the matrix A.

$\lambda[A]$. Eigenvalue of the matrix A.

$\max\{\gamma_i\}$: Maximum of the set of scalars γ_i.

A^{-1} · Inverse of matrix A.

$det(A)$: Determinant of matrix A.

A^* : Complex conjugate transpose of matrix A.

$||E||$ · The norm of matrix E i e., $||E|| = \lambda_M^{\frac{1}{2}}[E^*E]$.

$\forall i$: for all i.

$\{A_i, B_i\}$ · A set of systems.

(A_i, B_i) : A single system.

C^- : Open left half of the complex plane.

R^n : n dimensional Eucledian space.

$Re\ \sigma$: Real part of the complex number σ.

SQ : Simultaneous Quadratic (or Simultaneously Quadratically).

QS : Quadratic Stabilization.

SS : Simultaneous Stabilization.

CC . Controllable Companion.

LTI : Linear Time Invariant.

MIMO : Multi-input Multi-output.

Chapter 1

Introduction

The feedback mechanism has proved to be a powerful method in controlling the response of a system in a desired manner. One primary reason for using feedback control is to reduce the sensitivity of a system due to model uncertainties and external disturbances which can not be modeled exactly. The higher the sensitivity of the system, the larger the expected deviation in its output, which is undesirable. The search for ways to design a system insensitive to parameter variations, in fact, led to the discovery of electronic feedback amplifiers. In a stable closed loop system, the sensitivity is bounded (but not necessarily minimum). Hence one of the important qualitative problems in feedback control theory is to design a stable closed loop system despite the model uncertainties and perturbations.

In classical feedback theory, gain and phase margin were used as indices of the stability robustness against model perturbations. In the modern work on feedback stabilization theory, new problems were posed to consider explicitly the uncertainties and perturbations of the system being stabilized. These problems are popularly known as

1. Robust stabilization under unstructured uncertainties

2. Robust stabilization under structured (parametric) uncertainties and

3. Simultaneous stabilization

Unstructured uncertainty is specified as a norm bound of the model transfer function while parametric uncertainty is specified by a family of model transfer function whose structure is fixed but the parameters take values over specified intervals. The uncertainty of simultaneous stabilization is specified by a finite set of systems, and the problem is to

design a controller such that the closed loop system is stable, no matter which system in this finite set represents the plant.

1.1 Purpose of the Thesis

The purpose of this thesis is to study a special variety of the linear simultaneous stablization problem called the Simultaneous Quadratic (SQ) stabilization problem. This thesis is aimed at determining the parametric structures of finite number of state space models of control systems which can be stabilized by a single controller. We show the usefulness of these results for the stabilization of special classes of time varying systems as well as for the stabilization of uncertain time delay systems. We include case studies as applications of these results to (1) Controller design for robot manipulator for tracking purpose and (2) non-switching controller design for uncertain aircraft model.

1.2 Simultaneous Stabilizing Controller Design

To motivate the problems of this thesis, we consider a brief review of the simultaneous stabilizing controller design problem in control theory. Simultaneous stabilization problem is important to many practical situations in control theory. In practice a plant undergoes changes in several ways. For instance, a plant with a nonlinear model may have multiple operating points. The linearized dynamics of the plant at these points is thus different. Different operating points also arise due to hardware and software failures, most importantly sensor failures. In all the above cases, fixed stabilizing controller for different operating points is a desirable choice.

Another important application of simultaneous stabilization problem is to tackle system uncertainty due to uncertain physical parameters or limited information about the plant model. The uncertain physical parameters may consist of masses, inertias, spring constants, reaction rates, aerodynamic coefficients etc., that are required in mathematical descriptions of the plant. Linearization, model order reduction and neglected coupling terms are some of the examples of model inaccuracy Any model, therefore, is at best an approximation of reality Of course, the specific description of the source of uncertainties depends on the physics and the engineering of the particular system in question. Thus an important requirement to be satisfied by a controller is the invariance of the closed loop stability

property under model uncertainties.

In view of above considerations, the simultaneous stabilization problem is relevant in the following control problems.

- Design of a fixed stabilizing controller for uncertain system.
- Controller design with integrity, i.e., controller continues to operate in the presence of hardware and software failures.
- Single stabilizing controller design of
  - An aircraft for varying altitude and speed.
  - A power system for varying load conditions.
  - A missile for varying inertia and altitude.

1.2.1 Previous Approaches to Simultaneous Stabilization Problem

The simultaneous stabilization (SS) problem has a long history Several important results are reported on this problem in [57, 71, 23, 24] using the systematic approaches of factorization theory [72]. In [71], Vidyasagar and Viswanatham derived the existence of simultaneous stabilizing controller for the case of two plants. Roughly speaking, two plants are simultaneously stabilizable iff their difference plant is stabilizable by a stable controller. It was also shown that simultaneously stabilizing $(N+1)$ plants is equivalent to the simultaneous stabilization of N plants with a stable controller. In [81], Youla *et al.* studied the stabilization of a single plant by a stable controller using the property popularly known as *parity interlacing property* (p.i.p) A plant is said to satisfy p i.p. if between any two real right half plane zeros there exists even number of right half plane poles. So using p.i.p. condition, one can completely solve the simultaneous stabilization problem for two plant case. However, the problem of simultaneous stabilization for 3 or more plant cases is still open.

1.2.2 Quadratic Stabilization

Another approach to simultaneous stabilization problem can be the efficient use of Lyapunov stability theory Lyapunov stability theory is conventionally used more for the

analysis of linear and nonlinear, time invariant and time varying systems than for control system design. In recent times, robust controller design problem for uncertain system is addressed using Lyapunov stability theory. Also a concept called Quadratic Stabilization (QS) is introduced in modern control literature for studying stabilization of an uncertain system represented by the state space description. The basic philosophy of quadratic stabilization is to ascertain the existence of a single quadratic Lyapunov function to test the stability property of the uncertain system. An advantage of this approach is that these results can be applied for the stabilization of linear time varying systems in addition to the uncertain LTI systems. Consider a linear time varying uncertain system

$$\dot{x} = A(r(t))x + B(s(t))u = (A_0 + \Delta A(r(t)))x + (B_0 + \Delta B(r(t)))u \tag{1.1}$$

where $\Delta A(r(t))$, $\Delta B(r(t))$ represent the system uncertainty and (A_0, B_0) is the given **nominal** system model. The uncertain parameter vectors $r(t)$ belongs to some prescribed bounded set The problem of quadratic stabilization is to find a feedback controller such that the closed loop system is stable with a fixed (uncertainty independent) quadratic Lyapunov function. In this direction, important results are reported in [42, 4, 35, 54, 68, 51]. In [4], Barmish gave the necessary and sufficient conditions for the quadratic stabilizability and also gave the synthesis procedure for a nonlinear feedback controller. But these conditions are not easy to check.

The Riccati equation approach to the stabilization of uncertain systems has been developed by many authors [35, 54]. In this direction, attempt has been made to solve linear quadratic stabilization problem where control law is searched in the class of linear feedbacks. Along this line, Khargonekar *et al.* [35] have given some interesting results for the QS They consider the following norm bounded time varying uncertainty

$$[\Delta A(t)\ \Delta B(t)] = DF(t)E$$

where D, E are given constant matrices and $F(t)$ is such that $F^*(t)F(t) \leq I$(identity matrix). Then the existence of static state feedback controller for QS problem is associated with the existence of a positive definite solution of a parametric Riccati equation. But it is not immediately clear for which class of uncertain systems a stabilizing controller will

indeed exist. However, there are a number of results which consider the QS of the matched uncertain system (the perturbation ΔA and ΔB belong to the range space of nominal input matrix B_0). The uncertain system (1.1) is said to satisfy **matching condition** if the following conditions hold

$$\Delta A(r(t)) = B_0 D(r(t)), \quad \Delta B(r(t)) = B_0 E(r(t))$$

where $||E(r(t))|| < 1$ for all possible uncertainties $r(t)$.
Under this assumption, it is known in the literature that the **matched** uncertain system is quadratically stabilizable [32, 49]. Hence, in order to solve QS problem completely, an important aspect that remains to be investigated, is to determine whether there exists a controller for an uncertain system when the matching condition will not hold.

K. Wei [77] considered a single input time varying uncertain system where each uncertain element varies independently. Then he defined certain geometrical pattern called antisymmetric stepwise configuration, by augmenting system matrix $A(r(t))$ and input matrix $B(r(t))$ which were used for the QS of uncertain system by state feedback controller. In this case, it is assumed that the system must contain fixed number (equal to the system order) of sign invariant entries in proper locations.

Deviating from the norm bounded uncertainty representation, Geromel *et al.* [6] considered parametric uncertain system defined in convex bounded uncertainty domain. Consider a linear uncertain system described by

$$x = Ax + Bu \tag{1.2}$$

where A and B are matrices belonging to the uncertainty domains D_a and D_b respectively. The uncertainty domains D_a and D_b are defined as follows :

$$D_a = \{A : A = \sum_{i=1}^{N} \mu_i A_i, \ \mu_i \geq 0, \ \sum_{i=1}^{N} \mu_i = 1\}$$

$$D_b = \{B . B = \sum_{j=1}^{M} \alpha_j B_j, \ \alpha_j \geq 0, \ \sum_{j=1}^{M} \alpha_j = 1\}.$$

The matrices $\{A_i\}$ and $\{B_i\}$ represent the extreme points of the the convex sets D_a and D_b respectively The collection of all extreme plants $\{A_i, B_j\}$ for $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, M$ corresponds to a set of linear systems. The quadratic stabilization problem of uncertain system (1.2) is posed as a joint search problem for a quadratic Lyapunov

function and for a single state feedback gain for the collection of all extreme plants. Then authors proved that the convex bounded uncertain system is QS by a linear state feedback control law iff there exists a positive definite matrix W and a matrix $Z \in R^{m \times n}$ such that for all $i = 1, 2, \ldots N$ and $j = 1, 2, \ldots M$, the following matrix inequalities hold

$$A_i W + W A_i^* < B_j Z + Z^* B_j^*. \tag{1.3}$$

These matrix inequalities are linear with respect to two unknown matrices (one of which must be a positive definite matrix) By exploiting this linearity structures, they used linear programming to solve the QS problem. It is noticed that the above matrix inequalities are derived in terms of unknown matrices W and Z in addition to the given state space parameters $\{A_i\}$ and $\{B_i\}$. However, in practice, it is always advisable to derive the existence conditions in terms of given matrices A_i and B_i

1.2.3 Other Approaches to the Stabilization of Uncertain Systems

In this section, a few more approaches to the design of robust controller are briefly discussed Two general kinds of measure used in the literature to quantify the uncertainty of the plant are unstructured and structured uncertainty In the unstructured uncertainty representation, the perturbed plant is represented by the transfer matrix $G(s) = G_0 + \Delta G(s)$ where G_0 is the nominal transfer function. It is also assumed that for any proper stable transfer function $r(jw)$, $||\Delta G(jw)||_\infty \leq |r(jw)|$ for all frequency w where $||\Delta G||_\infty$ is the maximum singular value of ΔG. Two types of problem arises in this framework, namely analysis and design. In the analysis problem, given a stabilizing controller, one studies the extent of uncertainty to which the perturbed system will remain stable. More specifically, suppose $C(s)$ stabilizes the nominal plant, then the perturbed system will remain stable iff [18]

$$||C(I + G_0 C)^{-1}\, r(jw)||_\infty \leq 1 \; \forall \, w.$$

In the design problem, given a norm bound on the perturbation $\Delta G(jw)$, one determines whether there exists a robust stabilizing controller. However, this type of norm bounded uncertainty with no additional restriction on ΔG consists of a very large class of perturbations, and the controller design based on this often leads to a conservative design. In this context one may recall that several scientists such as Routh, Hurwitz, Bode. Nyquist and

several others, attempted to solve the stabilization problem. In those days the controller design was based on the assumed nominal model of the system to be controlled. The main focus was to find a stabilizing controller for the nominal plant, with no consideration of system uncertainties.

In many practical processes, the perturbations have a special form because they may originate from the variations of physical parameters. In such perturbations, which are called structured uncertainty, the transfer matrix or the state space model depends on the real uncertain parameter vector p $(= p_0 + \Delta p)$ with nominal value p_0 and the range of perturbation Δp. The measure of robustness, known as stability robustness margin, is then defined by $||\Delta P||$ for which closed loop stability will be preserved. The Routh-Hurwitz criterion, the Kharitinov theorem and related results [36, 9] are proved to be valuable application tools for the design of robust feedback stabilizing controller. These results are used to develop robust stability criteria using the uncertain characteristics polynomial of the system. For more details of the existence conditions of stabilizing controller see the references [7, 5, 76, 75]. However, in this direction, most of the computational algorithms to design controller are developed based on the assumptions that the the characteristic polynomial coefficient is an affine function of the controller gain. It is easily verified that this indeed is the case for all single input systems [5, 75]. However, for multi input systems, this assumption is difficult to satisfy [5].

Ackermann [1] used a graphical method to map a stability region (a region in the open left half of the complex plane) onto the space of state feedback gains for each plant. Then the intersection of all the mapped regions in the gain space gives the desired solution. His method is not easily applicable when there are more than two gain parameters Schemitindorf *et al.* [59] derived a sufficient condition for the existence of a state feedback controller which stabilizes a set of single input systems. They also gave controller design algorithm when the sufficient conditions are satisfied.

1.3 Motivation and Problem Considered in this Thesis

In general. two approaches are broadly used to solve robust stabilization problems, namely, the frequency domain approach and the time domain approach. In this thesis the latter

approach is followed. In the time domain approach, the notion of Lyapunov stability is adopted to solve the simultaneous stabilization problem for a finite number of linear systems using a constant feedback controller. In this direction. the results of Geromel *et al.* [6] give fairly general conditions for solving the stabilization of convex bounded uncertain systems. However, the above conditions are not constructive, i.e., there is at present no constructive algorithm to decide whether the matrix inequalities (1.3) can be solved. In other words, the question of existence of the stabilizing state feedback controller is still unresolved for the systems (1.2). So there is a need to derive the existence conditions as well as the tractable computational algorithm for the SS problem. For this purpose, we first consider the following set of systems described by state space systems for $i = 1, 2, \ldots, l$

$$\dot{x} = A_i x + B_i u, \quad y = Cx \tag{1.4}$$

where x is state vector, u is input and y is output vector. Next, we consider the following mathematical questions.

- Given a set of LTI system described by $\{A_i, B_i\}$. what are the conditions (parametric structures) on A_i and B_i for which there exists a single controller gain K such that all the eigenvalues of each closed loop matrix have negative real part, i.e , $\lambda(A_i - B_i K) \in C^-$? This problem corresponds to the design of static state feedback controller for simultaneous stabilization problem.

- Given a set of LTI system described by $\{A_i, B_i, C\}$, what are the structures of A_i and B_i and C for which there exists a single controller gain L such that $\lambda(A_i - B_i LC) \in C^-$. This problem corresponds to the design of a single static output feedback controller for simultaneous stabilization problem.

To give a partial answer of the above questions, we adopt an approach along the line of quadratic stabilization. This approach will be referred as Simultaneous Quadratic (SQ) stabilization (the formal definition will be given in chapter 2). The problem of simultaneous quadratic stabilization is to find a linear constant feedback gain for a finite number of LTI systems such that the eigenvalues of each closed loop matrix will lie in the open left half of the complex plane. Furthermore, the stabilizing feedback control law is required to ensure that each closed loop system is asymptotically stable with a common quadratic Lyapunov function. .In other words, SQ stabilization problem involves the design of a single state feedback control law $u = -Kx$ and a single positive definite matrix P such

that the derivative of a common quadratic Lyapunov function $V(x) = x^*Px$ is negative definite along the solution of each closed loop[1] system (1.4). Unlike the norm bounded time varying uncertainty representation of QS problem, the SQ stabilization problem is defined for a finite collection of LTI systems. Furthermore, no concept of nominal system is used in SQ stabilization problem. With this motivation, we pose the following questions

- What are the structures of $\{A_i, B_i\}$ (those beyond the matched uncertainty case) that are SQ stabilizable[2](i.e., the stabilizing state feedback controller exists)?
- What are the conditions for output feedback versions of SQ stabilization?
- How can these results be extended to the SQ stabilization of time delay systems?
- Whether and how these results would help to find a stabilizing controller for time varying situation?

In this thesis we analyze the SQ stabilization problem and answer the above questions.

1.3.1 Organization of the Thesis

The content of the present thesis is divided into different chapters as given below. A brief discussion of literature is also included in the introduction part of each chapter.

In chapter 2, we first define the problem statement of the SQ stabilization of finite number of LTI systems $\{A_i, B_i\}$ by state feedback controller. By assuming fixed B matrix, the solution of SQ stabilization problem is derived in terms of the solvability of a set of matrix inequalities. The next question considered is as to what the structures of $\{A_i\}$ and $\{B_i\}$ matrices should be for which SQ stabilization problem is solvable. Towards this goal, a set of single input second order systems is considered and the necessary and sufficient conditions for the SQ stabilization of these systems are derived. A set of matched uncertain systems (the formal definition will be given later) are shown to be SQ stabilizable As a corollary to this result, it is also shown that if a set of systems are simultaneously transformable into controllable companion form, then they also satisfies matching conditions. Next it is proved that a set of systems which are simultaneously transformable into

[1]After plugging the state feedback control law $u = -Kx$ in (1.4), the closed loop systems are represented by $x = (A_i - B_iK)x$

[2]In this thesis, the abbreviation SQ is used for the words 'Simultaneous Quadratic' or 'Simultaneously Quadratically' and this will be understood from the context.

Hessenberg form, are also SQ stabilizable. This class of systems is larger than the matched uncertain systems. Using the realization $\{A_i, B_i\}$ and under some technical assumptions on $\{B_i\}$ matrices, an algorithm for computing a state feedback controller of SQ stabilizable systems is outlined.

In chapter 3, we introduce two new family of systems and refer them as (1) partially commutative and (2) partially normal systems. These family of systems are defined under certain assumptions on the state space parameters $\{A_i\}$ and $\{B_i\}$. The existence of a state feedback controller for the SQ stabilization problem are then ensured for these classes of systems. Next, a more general class of systems is also identified for which SQ stabilization problem is solvable These systems exhibit uncertainties different from those matched uncertain systems.

To implement the state feedback controller, we require the knowledge of all state variables. However, in practice it is not always possible to measure all the states. So the design feedback control law using measurable outputs is a desirable choice. For this purpose, we consider the SQ stabilization problem using static output feedback controller. First, we derive the general existence conditions in terms of a set of matrix inequalities. Then different classes of systems are identified for which SQ stabilization problem is solvable by output feedback controller. These aspects are discussed in chapter 4.

In chapter 5, the results of chapter 2 and chapter 3 are extended for time delay systems and also for time varying systems We begin this chapter by discussing the asymptotic stability of linear state delay systems using quadratic Lyapunov functional Then the conditions for simultaneous quadratic stabilizability of a set of time delay systems using state feedback controller is derived. Different set of delay systems are then identified which are stabilizable by a single state feedback controller. In the second part of this chapter, we discuss the exponential stability of time varying systems. Different class of time varying systems are reported which are stabilizable by a constant state feedback controller

We consider the applications of SQ stabilization problem in chapter 6 Two practical examples are considered for this purpose. First example shows the design of a single stabilizing controller for an uncertain aircraft model The second example concerns with the design of tracking control law for a two link robot manipulator. Here the specific purpose of the control law is that the joint angle will track the prespecified reference trajectories.

Chapter 7 concludes the overall thesis.

Chapter 2

Simultaneous Quadratic Stabilization Problem

For a convex set of parameterized family of state space model, the quadratic stabilizability by linear state feedback control is equivalent to the solvability of a finite number of matrix inequalities as given in [6] As stated earlier in chapter 1, a numerical procedure using linear programming has been adopted to solve these inequalities [6] The work in this chapter is motivated by the lack of existence conditions of the solution of these matrix inequalities This chapter is devoted towards obtaining those structures of the state space parameters of a finite set of systems for which SQ stabilization problem is solvable (the formal definition will be given later).

The chapter is organized as follows . In the first section 2.1, we briefly state the quadratic stabilizability results developed in [6]. Next we define the SQ stabilization problem and carry out a preliminary analysis and also develop background results for the sequel Section 2.3 is then devoted to the solution of the SQ stabilization problem for special cases Finally, the solvability of SQ stabilization problem for special structures of the state space system parameters is investigated.

2.1 Matrix Inequalities Derived in [6]

Consider a uncertain system described by the following state space model

$$\dot{x} = Ax + Bu \tag{2.1}$$

where matrices A and B belong to the convex sets D_a and D_b respectively. The uncertainty domains D_a and D_b are defined as follows :

$$D_a = \{A : A = \sum_{i=1}^{N} \mu_i A_i,\ \mu_i \geq 0,\ \sum_{i=1}^{N} \mu_i = 1\}$$

$$D_b = \{B : B = \sum_{j=1}^{M} \alpha_j B_j,\ \alpha_j \geq 0,\ \sum_{j=1}^{m} \alpha_j = 1\}.$$

The matrices $\{A_i\}$ and $\{B_i\}$ represent the extreme points of the the convex sets D_a and D_b respectively. Each extreme plant (A_i, B_j) for $i = 1, 2, \ldots N$ and $j = 1, 2, \ldots, M$ corresponds to a LTI system. Then it is proved in [6] that the convex bounded uncertain system is QS by a linear state feedback control law iff there exists a positive definite matrix W and a matrix $Z \in R^{m \times n}$ such that for all $i = 1, 2, \ldots N$ and $j = 1, 2, \ldots M$,

$$A_i W + W A_i^* < B_j Z + Z^* B_j^*.$$

If these inequalities are solvable for matrices Z and W, then one can construct a robust stabilizing control law $u = -ZW^{-1}x$. It may be noted that these matrix inequalities are derived in terms of unknown matrices W and Z in addition to the given state space parameters $\{A_i\}$ and $\{B_i\}$. However, in practice it is always advisable to derive the existence conditions in terms of the given matrices A_i and B_i alone This observation has prompted us to define SQ stabilization problem for a set of LTI systems (not necessarily belonging to convex uncertain set). This is discussed below

2.2 Problem Formulation

In this section, the simultaneous stabilization by fixed state feedback controller is discussed For this purpose, we consider a set of LTI system described by state space equation,

$$\dot{x}^{(i)} = A_i x^{(i)} + B_i u^{(i)},\ i = 1, 2, \ldots l \qquad (2.2$$

where $x^{(i)} \in R^n$ is state vector, $u^{(i)} \in R^m$ is input vector. The matrices $A_i \in R^{n \times}$ and $B_i \in m \times n$ $(m \leq n)$ are real matrices respectively This set of systems will b denoted by $\{A_i, B_i\}$ whereas a single system will be denoted by (A_i, B_i). Each pair $(A_i, B_i$ is stabilizable. Based on this state space description, we introduce some definitions c simultaneous stabilization by static state feedback controller which are stated next.

Definition 2.1 : A set of systems $\{A_i, B_i\}$ is said to be Simultaneously Stabilizable (SS) by a state feedback controller if there exists a matrix K such that the the eigen values of each closed loop matrix $(A_i - B_i K)$ lies in the open left half of the complex plane.
Definition 2.2 : A single system (A_i, B_i) is said to be Quadratically Stabilizable (QS) by a state feedback controller if there exists matrices $W = W^* > 0$ and K satisfying the following matrix inequality[1]

$$(A_i - B_i K)^* W^{-1} + W^{-1}(A_i - B_i K) < 0.$$

Definition 2.3 : A set of LTI systems $\{A_i, B_i\}$ is said to be Simultaneously Quadratically (SQ) stabilizable by a state feedback controller if there exists matrices $W = W^* > 0$ and K such that the following matrix inequalities hold

$$(A_i - B_i K)^* W^{-1} + W^{-1}(A_i - B_i K) < 0 \;\; \forall\, i.$$

2.2.1 Discussion of the Definitions and Implications

Based on the definition 2.3, the first problem considered in this thesis is stated as follows.
SQ stabilization problem : Find the different structures of matrices A_i and B_i such that a set of systems $\{A_i, B_i\}$ are SQ stabilizable by state feedback controller.

Before proceeding further, the following well known result is stated based on the notion of Lyapunov stability of homogeneous LTI system $\dot{x} = Ax$ where x is the state vector.

Lemma 2.1 *The matrix A (associated with a LTI system $\dot{x} = Ax$) has all eigen values in the open left half of complex plane iff there exists a matrix $P = P^* > 0$ such that either of the following inequality holds*
(I) $A^*P + PA < 0$
or
(II) $AW + WA^* < 0$ *where* $W = P^{-1}$. □

In such situation, the derivative of the quadratic Lyapunov function $V(x) = x^*Px$ will be negative along the solution of $x = Ax$. The matrix P will be called Lyapunov matrix[2]. This lemma will be useful in designing a state and output feedback controller for a collection of LTI systems.

[1] The matrix W^{-1} is deliberately used to simplify the results derived latter.
[2] The matrix $W = P^{-1}$ will also be called Lyapunov matrix.

In the light of lemma 2.1, it is observed that if a set of systems is SQ stabilizable, then the stabilizing controller is given by $u^{(i)} = -Kx^{(i)}$ and the corresponding quadratic Lyapunov function for each system is $V(x^{(i)}) = (x^{(i)})^* W^{-1} x^{(i)}$.

It may also be noted that if a single system (A_i, B_i) is stabilizable then it is also quadratically stabilizable in the light of lemma 2.1. However, simultaneous stabilization of a set of systems (as given definition 2.1) does not necessarily mean that they are also SQ stabilizable. But one advantage of SQ stabilization will be that one can give some existence condition as well as a numerical algorithm.

2.2.2 Coordinate Dependence of SQ Stabilization Problem

Before we progress further towards the solution of SQ stabilization problem, we consider an important fact. Note that the SQ stabilization problem as stated above depends on the state space representation of the system to be stabilized, and the solution (i.e., the control law $u^{(i)} = -Kx^{(i)}$) depends on which state space coordinates are used for feedback. In other words, if a different state space representation is used for system representation and the state feedback is taken with respect to the new state space coordinates, then a non-stabilizable family may become stabilizable. To illustrate this fact consider the following example.

Example 2.1 : Consider the following two 2nd order state space models of the systems to be stabilized

$$\dot{x}^{(i)} = A_i x^{(i)} + B u^{(i)}, \ i = 1, 2 \tag{2 3}$$

where $A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$ and $B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Consider the problem of obtaining control laws $u^{(i)} = -Kx^{(i)}$, $i = 1, 2$ with a single feedback gain matrix $K = [k_1 \ k_2]$ such that each closed loop matrix $A_i - B_i K$ for $i = 1, 2$ is stable, i.e., all the eigen values of $(A_i - B_i K)$ are in the open left half plane. The characteristic polynomials of the closed loop matrices $A_1 - B_1 K$ and $A_2 - B_2 K$ are $s^2 + k_2 s + k_1$ and $s^2 + k_2 s - k_1$ respectively. Now using Routh-Hurwitz criterion, it can be concluded that

(a) $A_1 - B_1 K$ is stable for $k_1 > 0$, $k_2 > 0$ and

(b) $A_2 - B_2 K$ is stable for $k_1 < 0$, $k_2 > 0$.

This is impossible. So there does not exist a **single** matrix K such that the matrix $A_i - B_i K$ is stable (i.e , all eigenvalues of $(A_i - B_i K)$ are in the open left half plane) for $i = 1, 2$. Hence they are also not SQ stabilizable.

Now consider two new state variable vectors defined by $x^{(i)} = T_i y^{(i)}$ for $i = 1, 2$ where $T_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $T_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Using these state space transformations, we get the following two new systems

$$\dot{y}^{(i)} = A_{ni} y^{(i)} + B_{ni} u^{(i)}, \; i = 1, 2 \tag{2.4}$$

where $A_{n1} = T_1^{-1} A_1 T_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $A_{n2} = T_2^{-1} A_2 T_2 = A_{n1}$ and

$B_{n1} = T_1^{-1} B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $B_{n2} = T_2^{-1} B_2 = B_{n1}$

Now we see that using a constant matrix $K = [1 \; 1.732]$, each system is stablizable by control law $u^{(i)} = -K y^{(i)}$ for $i = 1, 2$. It is also observed that there exists a matrix $W = \begin{bmatrix} 0.866 & -0.5 \\ -0.5 & 0.866 \end{bmatrix}$ such that the following inequality holds for $i = 1, 2$

$$(A_i - B_i K)^* W^{-1} + W^{-1}(A_i - B_i K) < 0.$$

Thus in the new state space coordinates, $\{A_{ni}, B_n\}$ are SQ stabilizable. The above example shows that when state space coordinates of the set of systems $\{A_i, B_i\}$ are given, then the state space coordinates are fixed and the control law is to be chosen from the very same physical state variables. □

In view of the above observations, we shall assume henceforth that the state variables are fixed for all the systems and denoted by the same notation i.e., $x^{(i)} = x$ for all $i = 1, 2, \ldots, l$. Using the notation $u^{(i)} = u \; \forall \; i$, the system represented by (2.1) is given by

$$\dot{x} = A_i x + B_i u, \; i = 1, 2, \ldots l \tag{2.5}$$

The SQ stabilization problem is restated as follows.

Definition 2.4 : A set of LTI systems $\{A_i, B_i\}$ defined in (2.5) is said to be simultaneously quadratically stabilizable by a state feedback controller $u = -Kx$ if there exists a matrix $W = W^* > 0$ such that the following inequalities hold

$$(A_i - B_i K)^* W^{-1} + W^{-1}(A_i - B_i K) < 0 \; \forall \; i.$$

2.2.3 Problem Statement

Based on the definition 2.4, the statement of stabilization problem is as follows.

SQ stabilization problem : Find the different structures of matrices A_i and B_i such that a set of systems $\{A_i, B_i\}$ are simultaneously quadratically stabilizable by a state feedback controller.

Observe that SQ stabilization problem involves the design of a **single** state feedback controller $u = -Kx$ such that the derivative of a common quadratic Lyapunov function $V(x) = x^* W^{-1} x$ is negative definite for the solution of each closed loop systems.

2.3 Preliminaries

In this section, we consider a simplified SQ stabilization problem of $\{A_i, B_i\}$ as stated above, in which input matrix B_i are same for each system. However, later in section 2.5, this restriction on input matrix will be removed. Now consider a family of LTI systems $\{A_i, B\}, i = 1, 2, \ldots, l$ described by

$$\dot{x} = A_i x + Bu. \tag{2.6}$$

We make the following assumptions about the set of systems $\{A_i, B\}$.

Assumptions 2.1 : The matrix B is full column rank and is of the form

$$B = \begin{bmatrix} 0 \\ I_m \end{bmatrix}$$

where I_m is $m \times m$ identity matrix. This assumption does not restrict generality, for even if B is not of full rank, then by redefining the inputs one can have a B matrix of full rank. If the B matrix does not conform to the above structure, then there always exists a **single** similarity transformation such that the above assumption will hold. To further clarify this point, consider a set of systems $\{A_i, B\}$ described in (2.5) where B is not of the above structure Let T be the state space transformation such that $T^{-1}B = \begin{bmatrix} 0 \\ I_m \end{bmatrix}$. The new system will be given by $\{A_{ni}, B_n\}$ where $A_{ni} = T^{-1} A_i T$ and $B_n = T^{-1}B$. Now suppose

there exists a matrix K and W such that the new systems $\{A_{ni}, B_n\}$ are SQ stabilizable, i.e., the following inequalities hold

$$(A_{ni} - B_n K)^* W^{-1} + W^{-1}(A_{ni} - B_n K) < \mathbf{0} \; \forall \, i.$$

After rewriting the above inequalities in terms of original set of systems $\{A_i, B\}$, we obtain

$$(A_i - BK_o)^* W_o^{-1} + W_o^{-1}(A_i - BK_o) < \mathbf{0} \; \forall \, i$$

where $K_o = KT^{-1}$ and $W_o = TWT^*$.
Hence if the set of systems $\{A_{ni}, B_n\}$ is SQ stabilizable then $\{A_i, B\}$ is also SQ stabilizable Similarly, it can be shown that if the set of systems $\{A_i, B\}$ is SQ stabilizable then $\{A_{ni}, B_n\}$ is also SQ stabilizable. Thus the above assumption does not lead to any loss of generality.

Assumption 2.2 : For the SQ stabilization problem, we assume that the state feedback control law $u = -Kx$ is obtained with K in the form $K = (\gamma/2)B^* W^{-1}$ where γ is a positive scalar and W is a positive definite matrix. The structure of matrix K is justified using the following lemma.

Finsler's Lemma : Let M be a given $n \times n$ symmetric matrix and let N be an $m \times n$ matrix such that

$$x^* M x < 0$$

for all nonzero vectors $x \in R^n$ such that $Nx = \mathbf{0}$. Then there exists a constant $\sigma > 0$ such that

$$M - \sigma N^* N < \mathbf{0}.$$

Proof : A proof of this lemma can be found in [35, 33, 21]. □

The justification of above assumption 2 2 is as follows. From definition 2.4, a set of systems $\{A_i, B\}$ is SQ stabilizable if there exists matrices $W = W^* > 0$ and K such that for any vector $x \in R^n$ the following hold

$$x^*[(A_i - BK)^* W^{-1} + W^{-1}(A_i - BK)]x < 0 \; \forall \, x \in R^n \text{ and } \forall \, i.$$

$$\Rightarrow x^* \, [A_i W^{-1} + W^{-1} A_i^*]x < 0 \; \forall \, i \text{ and } x \text{ such that } B^* x = 0.$$

The above condition is a set of matrix inequalities. By Finsler's lemma for each of the

above inequalities, there exists a scalar γ_i such that

$$x^*[A_i^* W^{-1} + W^{-1} A_i - \gamma_i W^{-1} B B^* W^{-1}] x < 0 \; \forall \; x \in R^n \tag{2.7}$$

$$\Leftrightarrow x^* W^{-1} [A_i W + W A_i^* - \gamma_i B B^*] W^{-1} x < 0 \; \forall x \in R^n$$

$$\Leftrightarrow x^*[(A_i - BK_i)^* W^{-1} + W^{-1}(A_i - BK_i)^*] x < 0 \text{ where } K_i = (\gamma_i/2) B^* W^{-1}.$$

The above inequality (2.7) is simplified and rewritten as a matrix inequality

$$[A_i W + W A_i^* - \gamma_i B B^*] < 0. \tag{2.8}$$

In this way, we can derive a set of matrix inequalities of the form 2.8. In other words, there exists a set of positive scalars $\{\gamma_i\}$ such that the following inequalities hold .

$$[A_i W + W A_i^* - \gamma_i B B^*] < 0 \;\; \text{for } i = 1, 2, \ldots, l \tag{2.9}$$

Next define a scalar $\gamma = \max\{\gamma_i\}$. Using this value γ and noting that the matrix BB^* is a positive semidefinite matrix, the following inequalities will be satisfied.

$$[A_i W + W A_i^* - \gamma B B^*] < 0 \; \forall \; i \tag{2.10}$$

Furthermore, if (2.10) holds, then in the light of lemma 2.1, $V(x) = x^* W^{-1} x$ is a quadratic Lyapunov function for each closed loop system (2.5) with control law given by $u = -Kx = -(\gamma/2) B^* W^{-1} x$ where $\gamma (> 0)$ is a scalar. Thus it has been shown that if a set of system is SQ stabilizable then one controller gain structure will be $K = (\gamma/2) B^* W^{-1}$ This completes the justification of choosing the controller gain structure. □

The following primary lemma gives a necessary and sufficient conditions for the SQ stabilization of a set of systems $\{A_i, B\}$.

Lemma 2.2 *A set of systems $\{A_i, B\}$ is SQ stabilizable iff there exists a scalar $\gamma > 0$ and a positive definite matrix $W > 0$ such that*

$$[A_i W + W A_i^* - \gamma B B^*] < 0 \; \forall \, i. \tag{2.11}$$

Proof : By definition 2.3, a set of systems $\{A_i, B\}$ is SQ stabilizable iff there exists a controller gain K and W such that

$$(A_i - BK)^* W^{-1} + W^{-1}(A_i - BK) < 0 \; \forall \, i.$$

Now using $K = (\gamma/2) B^* W^{-1}$, the above inequalities are simplified by pre- and post-multiplying both sides by the symmetric matrix W as

$$A_i W + W A_i^* - \gamma B B^* < 0 \; \forall \, i \quad \square$$

Application of the above lemma 2.2 follows in the next section.

2.4 Basic Inequalities of SQ Stabilization Problem

In this section, we develop matrix inequalities which facilitate the investigation of SQ stabilizable family $\{A_i, B\}$ Before going to actual discussion, the given matrices A_i are partitioned as follows.

$$A_i = \begin{bmatrix} A_{11}^i & A_{12}^i \\ A_{21}^i & A_{22}^i \end{bmatrix}. \tag{2.12}$$

where $A_{11}^i \in R^{n-m \times n-m}$, $A_{12}^i \in R^{n-m \times m}$, $A_{22}^i \in R^{m \times m}$

Similarly, the positive definite $(n \times n)$ symmetric matrix W appearing in the inequalities (2.11) is partitioned as

$$W = \begin{bmatrix} W_1 & W_2 \\ W_2^* & W_3 \end{bmatrix} \tag{2.13}$$

where $W_1 \in R^{n-m \times n-m}$, $W_2 \in R^{n-m \times m}$, $W_3 \in R^{m \times m}$.

The following theorem is stated for the SQ stabilization of $\{A_i, B\}$ This result will be useful for the design algorithm of state feedback controller.

Theorem 2.1 *A finite set of systems $\{A_i, B\}, i = 1, 2, \ldots, l$ is simultaneously quadratically stabilizable iff there exists a matrix $W_1 = W_1^* > 0$ and a matrix F such that the following inequalities hold*

$$(A_{11}^i + A_{12}^i F)W_1 + W_1(A_{11}^i + A_{12}^i F)^* < 0 \; \forall \; i. \tag{2.14}$$

Proof (Sufficiency) :

Suppose there exists a matrix $W_1 = W_1^* > 0$ and a matrix F such that the following inequalities holds,

$$[(A_{11}^i + A_{12}^i F)W_1 + W_1(A_{11}^i + A_{12}^i F)^*] < 0 \; \forall \; i.$$

In order to satisfy (2.11), for SQ stabilization, we need to find a positive definite matrix W and a scalar $\gamma > 0$ such that $\Phi_i, i = 1, 2, \ldots, l$ will be negative definite where Φ_i is defined by

$$\Phi_i = [A_i W + W A_i^* - \gamma B B^*]. \tag{2.15}$$

For this purpose we construct a positive definite matrix W in the following way. By using the matrices F and W_1, choose $W_2^* = F W_1$. Then one can select W_3 such that $W_3 - W_2^* W_1^{-1} W_2 > 0$ which ensures that $W = \begin{bmatrix} W_1 & W_2 \\ W_2^* & W_3 \end{bmatrix}$ is positive definite matrix. Such a choice of W_3 is always possible. For instance, choose $W_3 = \alpha I_m$ where $\alpha > 0$ and $\alpha > \lambda_M[W_2^* W_1^{-1} W_2]$ where $\lambda_M[.]$ is the maximum eigenvalue of matrix [.]

With the above matrices W and F, the matrix Φ_i is partitioned as

$\Phi_i = \begin{bmatrix} \phi_{11}^i & \phi_{12}^i \\ (\phi_{12}^i)^* & \phi_{22}^i \end{bmatrix}$ where $\phi_{11}^i \in R^{n-m \times n-m}$.

The different submatrices are defined as follows

$$\phi_{11}^i = (A_{11}^i + A_{12}^i F)W_1 + W_1(A_{11}^i + A_{12}^i F)^* \tag{2.16}$$

$$\phi_{12}^i = A_{11}^i W_2 + A_{12}^i W_3 + W_1(A_{21}^i)^* + W_2(A_{22}^i)^*. \tag{2.17}$$

$$\phi_{22}^i = A_{21}^i W_2 + A_{22}^i W_3 + W_2^*(A_{21}^i)^* + W_3(A_{22}^i)^* - \gamma I_m \tag{2.18}$$

In the above equations $F = W_2^* W_1^{-1}$

The matrix Φ_i will be negative definite if [38]

$$(I)\ \phi_{11}^i < 0 \text{ and } (II)\ \phi_{22}^i - (\phi_{12}^i)^* \, (\phi_{11}^i)^{-1} \, \phi_{12}^i < 0.$$

It is observed that the matrix ϕ_{11}^i is negative definite for all i by the hypotheses of theorem. Also note that ϕ_{11}^i and ϕ_{12}^i are independent of γ. So the matrices $\{\phi_{22}^i\}$ can be made negative definite by choosing a sufficiently large value of γ. Since ϕ_{11}^i and ϕ_{12}^i are independent of γ, we can choose γ such that

$$\phi_{22}^i - (\phi_{12}^i)^* \, (\phi_{11}^i)^{-1} \, \phi_{12}^i < 0 \ \forall\, i. \tag{2.19}$$

The value of γ is computed as follows. Using the expression of ϕ_{22}^i from (2.18), the above inequalities (2.19) are rewritten as

$$\gamma I_m > S_i - (\phi_{12}^i)^* \, (\phi_{11}^i)^{-1} \, \phi_{12}^i \ \ \forall\, i \tag{2.20}$$

where $S_i = A_{21}^i W_2 + A_{22}^i W_3 + W_2^*(A_{21}^i)^* + W_3(A_{22}^i)^*$.

Now, for each index i, choose a scalar γ_i such that

$$\gamma_i > \lambda_M[\, S_i - (\phi_{12}^i)^* \, (\phi_{11}^i)^{-1} \, \phi_{12}^i \,]$$

where $\lambda_M[.]$ is the maximum eigenvalue of matrix [ ]. In this way we choose a set of scalars $\{\gamma_i\}$.

Let the value of $\gamma = \max\{\gamma_i\}$, $i = 1, 2, \ldots, l$. For this value of γ and W, the matrix Φ_i, defined in (2.15), will be negative definite for each $i = 1, 2, \ldots, l$. The controller gain is then given by $K = (\gamma/2)B^* W^{-1}$.

Necessary : Suppose, there exists a scalar $\gamma > 0$ and a matrix $W > 0$ such the following matrix inequality holds for each i :

$$\Phi_i = [A_i W + W A_i^* - \gamma B B^*] < 0 \tag{2.21}$$

Substituting $B = \begin{bmatrix} 0 \\ I_m \end{bmatrix}$ and partitioning $W = \begin{bmatrix} W_1 & W_2 \\ W_2^* & W_3 \end{bmatrix}$, the matrix Φ_i are partitioned as $\Phi_i = \begin{bmatrix} \phi_{11}^i & \phi_{12}^i \\ (\phi_{12}^i)^* & \phi_{22}^i \end{bmatrix}$. By using $F = W_2^* W_1^{-1}$, the different submatrices are

defined as in (2.16-2.18). Since the matrix Φ_i is negative definite for each i, so matrix ϕ_{11}^i is also negative definite for each i Hence the proof. □.

Remarks 2.1 : In theorem 2.1, we prove that a set of system $\{A_i, B\}$ is stabilizable by a state feedback controller iff there exists a matrix W_1 and F such that

$$\left(A_{11}^i + A_{12}^i F\right) W_1 + W_1 \left(A_{11}^i + A_{12}^i F\right)^* < 0 \, \forall \, i.$$

However, to verify this condition, the following steps should be followed

- Calculate a matrix F such that for each i, the matrix $(A_{11}^i + A_{12}^i F)$ is Hurwitz[3], i.e., all eigen values of $(A_{11}^i + A_{12}^i F)$ lies in the open left half plane To exist such a matrix F, the necessary condition is that each pair (A_{11}^i, A_{12}^i) must be stabilizable.
- Calculate $W_1 = W_1^* > 0$ by solving the Lyapunov equations

$$\left(A_{11}^i + A_{12}^i F\right) W_1 + W_1 \left(A_{11}^i + A_{12}^i F\right)^* < 0 \, \forall \, i$$

So to verify the existence conditions in theorem 2.1, it is necessary that each system pair (A_{11}^i, A_{12}^i) must be controllable or stabilizable.

Next result will be stated for the stabilizability of a system comprising the pair (A_{11}^i, A_{12}^i) defined in (2.12).

Lemma 2.3 *If a system (A_i, B) is controllable (stabilizable) then the pair (A_{11}^i, A_{12}^i) (constructed from the submatrices of A_i) is also controllable (stabilizable)*

Proof : The details of the proof of this lemma is given in Appendix A. □

Based on the theorem 2.1, we derive the existence conditions in terms of the solution of reduced order[4] matrix inequalities (2.14) as compared to (2.11) But one needs to know the structures of $\{A_i, B\}$ such that they are SQ stabilizable. The rest of the chapter will deal with the characterization such a few classes of systems. Before going to the special class of systems, the results of SQ stabilization for a set of second order systems will be derived in the next section.

[3]Note that a matrix S is called Hurwitz if all its eigenvalues have negative real parts. The terminology arises from the fact that if all eigenvalues of S have negative real parts, then the characteristic polynomial of S is a Hurwitz polynomial.

[4]In (2.11), the order of matrix inequalities is $n \times n$ whereas in (2.14) the dimension of matrix inequalities is $n - m \times n - m$.

2.5 Stabilization of Simple Systems

In this section, we discuss the stabilization of a set of second order systems. For this purpose, consider a set of single input second order systems $\{A_i, B\}$ where matrix A_i and B are defined as follows :

$$A_i = \begin{bmatrix} a_{11}^i & a_{12}^i \\ a_{21}^i & a_{22}^i \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Theorem 2.2 *A finite set of second order systems $\{A_i, B\}, i = 1, 2, \ldots, l$ is simultaneously quadratically stabilizable iff either of the following conditions holds*
(I) a_{12}^i is sign invariant entries $\forall\, i$
or
(II) $a_{11}^i < 0 \;\forall\, i$.

Proof : Using theorem 2.1, the set of systems $\{A_i, B\}$ will be SQ stabilizable iff for any scalar F and $W_1 > 0$, the following inequalities hold

$$\left[\, \left(a_{11}^i + a_{12}^i F\right) W_1 + W_1 \left(a_{11}^i + a_{12}^i F\right)^* \,\right] < 0 \;\forall\, i.$$

Since W_1 is a positive scalar, the above inequalities are ~~are~~ equivalently written as

$$a_{11}^i + a_{12}^i F < 0 \;\forall\, i. \tag{2.22}$$

Now the above inequalities (2.22) will hold iff
(I) a_{12}^i is sign invariant entries $\forall\, i$
or
(II) $a_{11}^i < 0 \;\forall\, i$. □

2.6 Stabilization of Matched Systems

The concept of matched uncertain system is defined in the robust control literature [32, 49] for linear systems with time varying uncertain parameters. In this case, the perturbation of system matrix lies in the range space of input matrix[5] as described in chapter 1. However, in our context, we shall call a finite number of LTI systems $\{A_i, B\}$ as matched systems if

[5]Suppose an uncertain system is represented by $\dot{x} = (A_0 + \Delta A)x + B_0 u$. Then this system will satisfy matching condition if $\Delta A = B_0 D$ where D is any matrix of proper dimension

the submatrices $\{A^i_{11}\}$ and $\{A^i_{12}\}$ (as defined in (2.12)) are same for all the systems The formal definition is as follows :

Definition 2.5 : A set of LTI system $\{A_i, B\}$ will be called matched system if the submatrices $\{A^i_{11}\}$ and $\{A^i_{12}\}$ are same for all the systems.

Remarks 2.2 : The above definition has been made by noting the special structure of matrix $B = \begin{bmatrix} 0 \\ I_m \end{bmatrix}$. In fact for matched systems, the difference between any two matrices $(A_i - A_j)$ lies in the range space of B

Defining this, the SQ stabilization results of such systems can be stated as follows.

Theorem 2.3 *A finite set of matched systems $\{A_i, B\}, i = 1, 2, \ldots, l$ described in (2.5) is simultaneously quadratically stabilizable.*

Proof : To prove the above theorem we will consider the special structures of matrices A_i and B. Based on the definition 2.5, the matrices A_i and B for matched systems are defined as follows ·

$$A_i = \begin{bmatrix} A_{11} & A_{12} \\ A^i_{21} & A^i_{22} \end{bmatrix} \tag{2.23}$$

and $B = \begin{bmatrix} 0 \\ I_m \end{bmatrix}$ where $A_{11} \in R^{n-m \times n-m}$, $A_{12} \in R^{n-m \times m}$, $A^i_{22} \in R^{m \times m}$.

We will show that there exists a positive definite matrix W_1 and a matrix F for which condition given in theorem 2 1 is satisfied.

From the inequality (2.14) of theorem 2 1, we need to show that there exists an F and W_1 such that the following inequality is satisfied

$$[\,(A_{11} + A_{12}F)\,W_1 + W_1\,(A_{11} + A_{12}F)^*\,] < 0. \tag{2.24}$$

This is a single Lyapunov inequality. It has positive definite matrix solution W_1 iff $(A_{11} + A_{12}F)$ is Hurwitz matrix. Since (A_{11}, A_{12}) is a stabilizable pair (since (A_i, B) is stabilizable and using lemma 2.3), there exists non-trivial solutions F for which matrix $(A_{11} + A_{12}F)$ is Hurwitz By choosing such a matrix F, there exists a matrix W_1 such that (2 24) holds. The computation of a W_1 is done by solving the following Lyapunov equation

$$[\,(A_{11} + A_{12}F)\,W_1 + W_1\,(A_{11} + A_{12}F)^*\,] = Q_0$$

where Q_0 is any $(n-m \times n-m)$ negative definite matrix. The existence of matrices W_1 and F completes the proof of the theorem. □

As a special case of the theorem 2.3, it is proved that, if each matrix $A_i, i = 1, 2, \ldots, l$ is in block controllable companion form [62] with respect to the same matrix B then $\{A_i, B\}$ are SQ stabilizable.

Theorem 2.4 *A finite set of systems $\{A_i, B\}, i = 1, 2, \ldots, l$ in block controllable companion form is simultaneously quadratically stabilizable.*

Proof · To prove the above theorem we will consider the special structure of matrices A_i and B as given below,

$$A_i = \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 \\ 0 & 0 & I_m & \cdots & 0 \\ \vdots & \cdot & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I_m \\ -\alpha_0^i & -\alpha_1^i & -\alpha_2^i & \cdot & -\alpha_r^i \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21}^i & A_{22}^i \end{bmatrix} \tag{2.25}$$

and $B = \begin{bmatrix} 0 \\ I_m \end{bmatrix}$ where $r = n/m$ and $A_{11} \in R^{n-m \times n-m}$, $A_{12} \in R^{n-m \times m}$, $A_{22}^i \in R^{m \times m}$.

Note that for each system the matrices A_{11} and A_{12} are same It is also noticed that the pair (A_{11}, A_{12}) is a controllable pair (being themselves in block companion form). Using theorem 2.3, we can conclude that $\{A_i, B\}$ are SQ stabilizable. This completes the proof of the theorem. □

Remarks 2.3 : For single input case ($m = 1$), the matrix A_i of (2.25) will be in the familiar controllable companion (CC) form and is given by

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdot & 1 \\ -\alpha_0^i & -\alpha_1^i & -\alpha_2^i & \cdots & -\alpha_n^i \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21}^i & A_{22}^i \end{bmatrix} \tag{2.26}$$

This form will be referred in later stage for comparison purpose. Next we shall state the

stabilizability results for a set of systems which are simultaneously transformable into CC form

Corollary 2.1 *Consider a set of systems $\{A_i, B\}$ having the property that there exists a nonsingular matrix T for which $\{T^{-1}A_iT, T^{-1}B\}$ are in controllable companion form, then the family $\{A_i, B\}$ is simultaneously quadratically stabilizable.*

Proof : First compute matrix T for any one system. Then transform all given systems into CC form using matrix T. The proof directly follows from the result of theorem 2.4 and as follows.
Let $A_{ci} = T^{-1}A_iT$ and $B_c = T^{-1}B$ where (A_{ci}, B_c) are in CC form. Using the result of theorem 2.4, there exists matrices $W = W^* > 0$ and K such that

$$x^*[\ (A_{ci} - B_cK)^*W^{-1} + W^{-1}(A_{ci} - B_cK)\ ]x < 0 \text{ for any } x.$$

$$\Rightarrow x^*[\ (T^{-1}A_iT - T^{-1}BK)^*W^{-1} + W^{-1}(T^{-1}A_iT - T^{-1}BK)\ ]x < 0.$$

$$\Rightarrow x^*T^*[\ (A_i - BK_o)^*W_o^{-1} + W_o^{-1}(A_i - BK_o)\ ]Tx < 0 \text{ where } W_o^{-1} = (T^{-1})^*W^{-1}T^{-1}$$
and $K_o = KT^{-1}$ □

Remarks 2.4 : Suppose we are given a finite number of systems $\{A_i, B\}, i = 1, 2, \ldots, l$ and they are transformable into CC form using a matrix T. First transform all the systems into CC form and then design controller gain K for the transformed systems. So the actual controller gain to stabilize all the original systems $\{A_i, B\}$ is given by $K_o = KT^{-1}$.

2.6.1 The class of simultaneously transformable systems

In the last section we have proved that whenever a set of plants $\{A_i, B\}$, $i = 1, 2, \ldots, l$ is in controllable companion form there exits a single Lyapunov function for the SQ stabilizability of the whole set. We show here that the class of single input systems $\{A_i, B\}$, which are simultaneously transformable into CC form, is equal to the class of systems satisfying matching condition. Suppose we are given a controllable pair (A_1, B) and T be the transformation matrix such that $A_{c1} = T^{-1}A_1T, B_c = T^{-1}B$. Construct the matrix T as $T = [\ e_1, e_2, \ \ldots e_n\ ]$ as given in Wonham [79] using the characteristic polynomial of matrix A_1 Here the column vector e_n is nothing but the vector B itself. After defining T.

the following class of systems can be transformed into CC form :

$$A_\alpha = A_1 + T \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ d_1 & \dots & d_n \end{bmatrix} T^{-1}$$ where $d = [\, d_1, \dots, d_n \,]$ is a arbitrary n dimensional row vector. After simplification, A_α can be written as $A_\alpha = A_1 + B\Omega$ where Ω is any arbitrary row vector. So the perturbation matrix $(A_\alpha - A_1)$ lies in the range space of input matrix B and hence satisfies the matching condition.

2.7 Stabilization of Hessenberg Family

In subsection (2.6), it has been shown that the matched family of systems are simultaneously stabilizable. To enlarge this class of system, we consider a family of systems which will be referred as Hessenberg family. In this section, we shall discuss the stabilization of a set of single input systems which are in Hessenberg family For this purpose, we consider a set of n dimensional state space system $\{A_i(n), B(n)\}$ described as

$$\dot{x} = A_i(n)x + B(n)u. \tag{2.27}$$

where $x \in R^n$ and $u \in R$. In the above equation, the matrices $\{A_i(n)\}$ are assumed to be in Hessenberg form [41], i.e.,

$$A_i(n) = \begin{bmatrix} h^i_{11} & 1 & 0 & \dots & 0 \\ h^i_{21} & h^i_{22} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h^i_{n-1,1} & h^i_{n-1,2} & h^i_{n-1,3} & \dots & 1 \\ h^i_{n,1} & h^i_{n,2} & h^i_{n,3} & \dots & h^i_{n,n} \end{bmatrix} \in R^{n \times n} \tag{2.28}$$

and $B(n) = \begin{bmatrix} 0 & . & . & 0 & 1 \end{bmatrix}^* \in R^n$. In the above notation, the integer n indicates the dimension of the state space description of the system. It is noticed that the companion form of a matrix defined in (2.26) is a special case of the Hessenberg form. Now consider the following definition :

Definition 2.6 : A set of systems $\{A_i(n), B(n)\}$ will be called Hessenberg family if the matrices $\{A_i\}$ are in Hessenberg form.

Next we show that, if each matrix $A_i(n), i = 1, 2, \ldots, l$ is in Hessenberg form along with the same matrix $B(n)$ in (2.27) then they are SQ stabilizable. This class of system is also shown to be larger than the controllable uncertain systems which satisfy the matching condition.

Theorem 2.5 *A finite set of n dimensional systems $\{A_i(n), B(n)\}, i = 1, 2, \ldots, l$ in Hessenberg form is simultaneously quadratically stabilizable.*

Proof : We prove the theorem by the method of induction. The matrices used in the following steps are assumed to be compatible in dimension.

Step-1 : Consider a set of two dimensional system $\{A_i(2), B(2)\}$ in Hessenberg form defined as follows :

$$A_i(2) = \begin{bmatrix} h_{11}^i & 1 \\ h_{21}^i & h_{22}^i \end{bmatrix} \text{ and } B(2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Using theorem 2.1, the set of systems $\{A_i(2), B(2)\}$ will be SQ stabilizable iff for any scalar F and a $W_1 > 0$ the following inequalities hold

$$\left[\left(h_{11}^i + F\right) W_1 + W_1 \left(h_{11}^i + F\right)^* \right] < 0 \; \forall \, i. \tag{2.29}$$

To satisfy the above inequalities, there always exists an F such that

$$(h_{11}^i + F < 0) \; \forall \, i.$$

By choosing such an F, one can calculate $W_1 > 0$ from (2.29). This proves the existence conditions for the SQ stabilization of $\{A_i(2), B(2)\}$. The controller gain is constructed based on the steps followed in theorem 2.1 and as given below.

Choose $W_2^* = FW_1$ and W_3 such that $W_3 - W_2^* W_1^{-1} W_2 > 0$ which ensures that $W = \begin{bmatrix} W_1 & W_2 \\ W_2^* & W_3 \end{bmatrix}$ is positive definite matrix. Such a choice of W_3 is clearly always possible. Using this W matrix, there exists γ such that the following holds for all the systems (using (2.11)) :

$$A_i(2)W + W(A_i(2))^* - \gamma B(2)(B(2))^* < 0 \; \forall \, i. \tag{2.30}$$

So the required controller gain will be $K = -(\gamma/2)(B(2))^* W^{-1}$. Here we denote this controller gain and the Lyapunov matrix W, to be used in the next step, as follows :

$$K(2) = K \text{ and } W(2) = W.$$

From the above discussions, it is concluded that *any* finite collection of two dimensional systems $\{A_i(2), B(2)\}$ in Hessenberg form is SQ stabilizable. This result will be subsequently used in the following step.

Step-2 : Consider a set of three dimensional system $\{A_i(3), B(3)\}$ defined by :

$$A_i(3) = \begin{bmatrix} h_{11}^i & 1 & 0 \\ h_{21}^i & h_{22}^i & 1 \\ h_{31}^i & h_{32}^i & h_{33}^i \end{bmatrix} = \begin{bmatrix} A_i(2) & B(2) \\ A_{21}^i & A_{22}^i \end{bmatrix} \text{ and } B(3) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

In the above partition, matrices are $A_i(2) = \begin{bmatrix} h_{11}^i & 1 \\ h_{21}^i & h_{22}^i \end{bmatrix}$, $B(2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $A_{21}^i \in R^{1\times 2}$ and $A_{22}^i \in R$ Using theorem 2 1, the set of systems $\{A_i(3), B(3)\}$ will be SQ stabilizable iff for any matrix F and $W_1 > 0$ the following inequalities hold

$$[\,(A_i(2) + B(2)F)\,W_1 + W_1\,(A_i(2) + B(2)F)^*\,] < 0 \;\forall\, i. \tag{2.31}$$

It is noticed that the systems $\{A_i(2), B(2)\}$ are in Hessenberg form. So from the results of step-1, there exists an F and $W_1 > 0$ such that (2.31) holds. At this point, the matrices F and W_1 are taken from step-1 and are given by :

$$F = K(2) \text{ and } W_1 = W(2)$$

Now choose $W_2^* = FW_1$ and W_3 such that $W_3 - W_2^* W_1^{-1} W_2 > 0$ which ensures that $W = \begin{bmatrix} W_1 & W_2 \\ W_2^* & W_3 \end{bmatrix}$ is positive definite matrix. Using this W matrix, there exists γ such that the following holds for all the systems using (2.11) :

$$A_i(3)W + W(A_i(3)^*) - \gamma B(3)(B(3))^* < 0 \;\forall\, i. \tag{2.32}$$

So the required controller gain will be $K = -(\gamma/2)(B(3))^* W^{-1}$ From the above discussion, it is concluded that *any* finite collection of three dimensional systems $\{A_i(3), B(3)\}$ in Hessenberg form are SQ stabilizable.

Step-3 : It is now assumed that a set of $(n-1)$ dimensional systems $\{A_i(n-1), B(n-1)\}$ in Hessenberg family is SQ stabilizable. From theorem 2.1, it follows therefore that there exists an F and $W_1 = W_1^* > 0$ such that

$$[\,(A_i(n-1) + B(n-1)F)\,W_1 + W_1\,(A_i(n-1) + B(n-1)F)^*\,] < 0\;\forall\, i. \tag{2.33}$$

The problem in hand is to show that a set of n dimensional systems $\{A_i(n), B(n)\}$ is also SQ stabilizable. Towards that end, we partition the special structure of $A_i(n)$ matrices of (2.28) as

$$A_i(n) = \begin{bmatrix} h_{11}^i & 1 & 0 & \dots & 0 \\ h_{21}^i & h_{22}^i & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h_{n-1,1}^i & h_{n-1,2}^i & h_{n-1,3}^i & \dots & 1 \\ h_{n,1}^i & h_{n,2}^i & h_{n,3}^i & \dots & h_{n,n}^i \end{bmatrix} = \begin{bmatrix} A_i(n-1) & B(n-1) \\ A_{21}^i & A_{22}^i \end{bmatrix}$$

where $B(n-1) = \begin{bmatrix} 0 & \dots & 1 \end{bmatrix}^* \in R^{n-1\times 1}$, $A_{21}^i \in R^{1\times n-1}$ and $A_{22}^i \in R$.

With the help of theorem 2.1, it is concluded that the set of systems $\{A_i(n), B(n)\}$ is SQ stabilizable if (2.33) holds. The required controller gain for the SQ stabilization of $\{A_i(n), B(n)\}$ can be arrived at using similar arguments as stated in step-1 and step-2. This completes the proof of the theorem. □

Next we shall state the stabilizability results for a set of systems $\{A_i(n), B(n)\}$ which are simultaneously transformable into Hessenberg form.

Corollary 2.2 *Consider a set of systems $\{A_i(n), B(n)\}$ having the property that there exists a nonsingular matrix T for which $\{T^{-1}A_i(n)T, T^{-1}B(n)\}$ are in Hessenberg form. then the family $\{A_i(n), B(n)\}$ is simultaneously quadratically stabilizable.*

Proof : The proof is similar to corollary 2.1. □

Remarks 2.5 : It is noticed from (2.26) and (2.28) that the controllable companion form is a special case of Hessenberg form. So the class of systems which are simultaneously transformable into Hessenberg form is larger than the controllable matched uncertain systems. As an example consider a set of two systems $\{A_i, B\}$ which are not in companion

form defined as follows :

$$\dot{x} = A_i x + Bu,\ i = 1,2 \text{ where } A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Notice that the perturbation matrix $(A_2 - A_1)$ does not belong to the range space of matrix B and hence does not satisfy matching condition. But these two systems are in Hessenberg form. So they are SQ stabilizable by theorem 2.5.

Remarks 2.6 : The proof of the theorem 2.5 also gives a construction procedure to compute the controller gain as well as the Lyapunov matrix for the SQ stabilization of finite number of systems $\{A_i, B\}$ which belong to Hessenberg family.

Remarks 2.7 : The results of this section can also be extended to multi-input systems using block Hessenberg form. A set of systems $\{A_i, B\}$ in block Hessenberg form (see [67, 41]) is described as follows :

$$A_i = \begin{bmatrix} h^i_{11} & h_{12} & 0 & . & 0 \\ h^i_{21} & h^i_{22} & h_{2,3} & . & 0 \\ . & \vdots & \vdots & \vdots & \vdots \\ h^i_{k-1,1} & h^i_{k-2,2} & h^i_{k-1,3} & .. & h_{k-1,k} \\ h^i_{k,1} & h^i_{k,2} & h^i_{n-2,n-2} & .. & h^i_{k,k} \end{bmatrix},\ B = \begin{bmatrix} 0 \\ I_m \end{bmatrix}.$$

In the above matrix A_i, the block matrices $h_{j,j+1}$ are of dimension $t_j \times t_{j+1}$, $j \in \{1, 2, \ldots, k-1\}$ and have full row rank t_j. In this case, $t_1 \leq t_2 \leq . \ . \leq t_k = m$ and $t_1 + t_2 + \ldots + t_k = n$. Similar to the single input cases, it can be proved by the method of induction that a set of systems $\{A_i, B\}$ in block Hessenberg form are SQ stabilizable. These details are omitted.

2.8 Results For General Input Matrix Case

In the previous section, we discussed the SQ stabilizability conditions of $\{A_i, B\}$ by assuming that the input matrix B is fixed for all the systems. In this section, we shall partially remove this restriction. Consider a set of systems $\{A_i, B_i\}, i = 1, 2, \ldots, l$ which are described by the state equation

$$\dot{x} = A_i x + B_i u \tag{2.34}$$

In this thesis, the following assumptions are made for the input matrices of the systems $\{A_i, B_i\}$.

Assumption 2.3 : Each of the input matrices $\{B_i\}$ are of the form $B_i = BG_i$ where nonsingular matrix G_i is such that $(G_i + G_i^*) > 0 \; \forall \, i$.
The matrix B is also assumed as

$$B = \begin{bmatrix} 0 \\ I_m \end{bmatrix}.$$

Assumption 2.4 : The structure of static state feedback controller gain for SQ stabilization problem is assumed as

$$K = (\gamma/2)B^* W^{-1} \tag{2.35}$$

where γ is a positive scalar and W is a positive definite matrix.

The justification of these assumptions follows on similar lines as that for assumptions (2.1-2.2) (page 16) using Finsler's lemma and the SQ stabilization of $\{A_i, B\}$. It is noticed that the input matrices $\{B_i\}$ are restricted[6] here in the sense that they represent particular types of perturbations. However this class of matrices arises in certain practical applications which are considered in chapter 6.

Based on the above assumptions 2.3-2.4, the following result is stated for the stabilization of a set of systems $\{A_i, B_i\}$ This result is similar to lemma 2.3 and the simplified version of the definition 2.4.

Lemma 2.4 *A set of systems $\{A_i, B_i\}$ are simultaneously quadratically stabilizable iff there exists a scalar $\gamma > 0$ and a positive definite matrix $W > 0$ such that*

$$A_i W + W A_i^* - (\gamma/2) B (G_i + G_i^*) B^* < 0 \; \forall \, i \tag{2.36}$$

Proof : By definition, a set of systems $\{A_i, B_i\}$ is SQ stabilizable if there exists a controller gain matrix K and a $W > 0$ such that

$$(A_i - B_i K)^* W^{-1} + W^{-1}(A_i - B_i K) < 0.$$

Now using $K = (\gamma/2)B^* W^{-1}$ and multiplying both sides by the symmetric matrix W, we get

$$A_i^* W + W A_i^* - (\gamma/2)(B B_i^* + B_i B^*) < 0$$

$$\Leftrightarrow \quad A_i^* W + W A_i^* - (\gamma/2) B (G_i^* + G_i) B^* < 0 \; \square$$

[6]This type of uncertainty in input matrix also arises for matching condition with time varying uncertainty representation [59] and is discussed in chapter 1 (page 5). However, in the present case of finite family of systems, we are not imposing any norm type conditions.

Next we further simplify the above lemma in the form of following theorem.

Theorem 2.6 *A finite set of systems $\{A_i, B_i\}, i = 1, 2, \ldots, l$ is simultaneously quadratically stabilizable iff there exists a matrix $W_1 = W_1^* > 0$ and a matrix F such that the following inequalities hold*

$$(A_{11}^i + A_{12}^i F)W_1 + W_1(A_{11}^i + A_{12}^i F)^* < 0 \; \forall \; i.$$

Proof : The proof is similar to the theorem 2.1. The details are however given in Appendix B. □

Remarks 2.8 : Note that in theorems 2.1 and 2.6, the existence conditions for the SQ stabilization of a set of systems $\{A_i, B_i\}$ are same as that for $\{A_i, B\}$ with fixed B. However the controller gain will be different for the two cases.
In the light of the above remark, the SQ stabilization of $\{A_i, B_i\}$ can be stated for matched systems similar to theorems 2.3 and 2 5. This is discussed below in brief.

Definition 2.7 : A collection of LTI systems $\{A_i, B_i\}$, with assumptions 2.3, will be called matched systems if the submatrices A_{11}^i and A_{12}^i are same for all the systems.

Theorem 2.7 *A finite set of matched systems $\{A_i, B_i\}, i = 1, 2, \ldots, l$ is simultaneously quadratically stabilizable*

Proof : Using theorem 2.6, the proof of this result is similar to that of theorem 2.3 □
Similar results can be stated for the Hessenberg family of systems also.

2.9 Controller Design Algorithm

Based on theorem 2 1 and theorem 2.6, we present successive steps to compute static state feedback controller gain for the finite set of systems $\{A_i, B_i\}, i = 1, 2, \ldots, l$ defined in (2.5). In this case it is assumed that the family of systems is SQ stabilizable. So from theorems 2.5 -2.6, there exists a positive definite matrix W_1 and F such that

$$\left(A_{11}^i + A_{12}^i F\right) W_1 + W_1 \left(A_{11}^i + A_{12}^i F\right)^* < 0 \; \forall \; i.$$

Step-0 : This step is for the computation of F and W_1 such that above inequalities hold. To compute these matrices, the following should be done

- Compute a matrix F such that for each i, the matrix $(A_{11}^i + A_{12}^i F)$ is Hurwitz.
- Compute W_1 by solving the Lyapunov equations

$$\left(A_{11}^i + A_{12}^i F\right) W_1 + W_1 \left(A_{11}^i + A_{12}^i F\right)^* < 0 \; \forall \, i.$$

In this chapter, we discuss the existence conditions of SQ stabilization for matched and Hessenberg family of systems. Hence, using the proof of theorems 2.3, 2.6 and 2.5, the matrices F and W_1 are obtained for these class of systems. In the next chapter, we shall discuss the computation of matrices F and W_1 for the systems referred as (a) partially commutative and normal systems and (b) more general family of systems. The matrices F and W_1 are used in the next step of algorithm.

Step-1 : Using the matrices W_1 and F, compute Q_0^i such that $Q_0^i = (A_{11}^i + A_{12}^i F) W_1 + W_1 (A_{11}^i + A_{12}^i F)^*$ for each i. Here Q_0^i is $(n - m \times n - m)$ dimensional negative definite matrix.

Step-2 : Compute $W_2^* = F W_1$

Step-3 : Select W_3 such that $W_3 > W_2^* W_1^{-1} W_2$ to ensure $W = \begin{bmatrix} W_1 & W_2 \\ W_2^* & W_3 \end{bmatrix}$ be a positive definite matrix. Choose $W_3 = r + W_2^* W_1^{-1} W_2$ where r is a positive definite matrix.

Step-4 : Using (2.36), compute the value of γ, and the required control law is given by $u = -Kx = -(\gamma/2) B^* W^{-1}$.
The computation of γ, to satisfy (2.36), is simplified as follows.

Define $\Phi_i = A_i W + W A_i^* - (\gamma/2) B (G_i + G_i^*) B^* = \begin{bmatrix} Q_0^i & q_i \\ q_i^* & q_{2i} - (\gamma/2) I_m (G_i + G_i^*) \end{bmatrix}$ where

$q_i = A_{11}^i W_2 + A_{12}^i W_3 + W_1 (A_{21}^i)^* + W_2 (A_{22}^i)^*$ and
$q_{2i} = A_{21}^i W_2 + A_{22}^i W_3 + W_2^* (A_{21}^i)^* + W_3 (A_{22}^i)^*$.

Now the problem is to find a scalar γ such that the matrices $\{\Phi_i\}$ will be negative definite for each $i = 1, 2, \ldots, l$. Note that the matrix Q_0^i is a negative definite matrix defined in step -1. So matrices $\{\Phi_i\}$ will be negative definite if [38]

$$q_{2i} - (\gamma/2)(G_i + G_i^*) - q_i^*(Q_0^i)^{-1}q_i < 0 \;\forall\; i.$$

For each index i, compute a scalar γ_i such that the following condition holds :

$$\gamma_i > 2\lambda_M \left[(q_{2i} - q_i^*(Q_0^i)^{-1}q_i)(G_i + G_i^*)^{-1}\right]$$

The required γ is selected as the maximum of all the scalars $\{\gamma_i\}$, i.e.,

$$\gamma = \max\{\gamma_i\}.$$

Example 2.2 : In this example, a single stabilizing feedback controller is designed for a set of three systems $\{A_i, B\}, i = 1, 2, 3$ using the algorithm given in subsection 2.2. The system parameters are given below

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 2.375 & -0.25 & -0.875 \\ 2.75 & 0.5 & 0.25 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1.875 & -1.25 & 0.625 \\ 1.75 & -1.5 & 3.25 \end{bmatrix}$$

$B = [\,0\,1\,2\,]^*$. The matrix T for the system (A_0, B) is given by $T = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ -2 & 1 & 2 \end{bmatrix}$.

Using matrix T, one can transform all three systems into CC form. In the algorithm we use $f = [\,-2\;-3\,]$, $Q_0 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $r = 40$.

The matrices W and K, computed using $\gamma = 3000$, are given below

$W = \begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 0.5 & -0.5 \\ -0.5 & -0.5 & 42.5 \end{bmatrix}$ and $K = [\,75\;112.5\;37.5\,]$. The controller gain to stabilize the above three systems is $K_o = KT^{-1} = [\,-93.75\;-75\;56.25\,]$.

2.10 Conclusion

In this chapter, the simultaneous quadratic stabilization problem is formulated as the problem of solving a set of matrix inequalities (MI). If these MI are successively solved

then their solution can be used to obtain a state feedback stabilizing controller for a set of LTI systems. Towards this end, a set of system $\{A_i, B\}$ with fixed B is considered. Then an attempt is made to identify different class of systems for which SQ stabilization problem is solvable, i.e., the required matrix inequalities have a solution. Following results are derived in this direction. The existence condition of matched uncertain system is discussed in the context of SQ stabilization problem. Here the matching condition is reinterpreted in terms of the submatrices of system matrices $\{A_i\}$. This has been done by exploiting the structure of the input matrix B. As a special case of the matched uncertain system, it is shown that the class of system is SQ stabilizable if $\{A_i\}$ are simultaneously transformable into *Controllable Companion* form. We also characterize a class of systems which are simultaneously transformable into companion form. In continuation of the above results, it has also been proved that a collection of LTI systems is stabilizable by a single controller if they are simultaneously transformable into Hessenberg form. This class of systems is shown to be larger than the class of uncertain systems which satisfies the matching condition. So the simultaneous stabilizability results of this chapter (derived for finite number of LTI systems) are different from the existing robust stabilizability property of the matched uncertain systems. To accommodate the uncertainty of input matrix, we consider the SQ stabilization of $\{A_i, B_i\}$. Then under some technical assumptions on B_i, the existence conditions of the set $\{A_i, B_i\}$ are same as those for the set $\{A_i, B\}$. The controller design algorithm for this finite SQ stabilizable family of systems $\{A_i, B_i\}$ is also given.

The companion form or Hessenberg form of a matrix has been used in the various design algorithms in control theory such as pole placement, model order reduction etc [41]. In this chapter, the special structure of companion and Hessenberg matrices has been explored to partially solve the SQ stabilization problem by static state feedback controller.

Chapter 3

Stabilization of Partially Commutative and Normal Systems

In this chapter, two new classes of systems are introduced for which the simultaneous quadratic stabilization problem is solvable. The existence conditions and the controller design algorithm are proposed for this purpose. Any finite number of systems $\{A_i, B_i\}, i = 1, 2, \ldots, l$ of the form defined (2.5), are SQ stabilizable if the matrices $\{A_{11}^i\}$ are Hurwitz and commutative or normal In addition to these systems, one more general class of systems is introduced here which are stabilizable by a single feedback controller. These classes of systems are different from the matched uncertain systems. Hence, the simultaneous stabilizability results of this chapter are quite different from that of chapter 2.

The chapter is outlined as follows In section 3.1 the SQ stabilization problem of commutative and normal family is discussed In section 3.2, the controller design for a more general class of systems is derived.

3.1 SQ Stabilization of Commutative and Normal Systems

Consider the following family of LTI systems $\{A_i, B_i\}, i = 1, 2, \ldots, l$ which are described by the state space equation

$$x = A_i x + B_i u \tag{3.1}$$

where $A_i \in R^{n\times n}$ and $B_i \in R^{n\times m}$. As in the chapter 2 (page 32), it is assumed that each matrix B_i satisfies assumptions 2.3, i.e., $B_i = \begin{bmatrix} 0 \\ I_m \end{bmatrix} G_i$ and $(G_i + G_i^*) > 0$.
The matrices A_i are partitioned as

$$A_i = \begin{bmatrix} A_{11}^i & A_{12}^i \\ A_{21}^i & A_{22}^i \end{bmatrix} \tag{3.2}$$

where $A_{11}^i \in R^{n-m\times n-m}$ and $A_{22}^i \in R^{m\times m}$.

Next two lemmas show the existence of common Lyapunov matrix for a set of Hurwitz matrices. These results will be successively used to derive the existence conditions of the SQ stabilization problem

Lemma 3.1 *If a set of matrices $\{M_i\}, i = 1, 2, \ldots, l$ are Hurwitz and commutative pair wise then there exists a common Lyapunov matrix P such that the following holds :*

$$M_iP + PM_i^* < 0 \;\forall\; i = 1, 2, \ldots, l.$$

Proof : A proof of this result is given in [46] We are reproducing this proof from [46] for the sake of comparison For the ease of presentation, first consider two Hurwitz commutative matrices M_1 and M_2 Suppose P_0 is a positive definite matrix. Let P_1 and P_2 be the unique positive definite solutions of the Lyapunov equations

$$M_1P_1 + P_1M_1^* = -P_0 \tag{3.3}$$

$$M_2P_2 + P_2M_2^* = -P_1. \tag{3.4}$$

Substituting for P_1 from (3.4) into (3.3) and using commutativity of M_1 and M_2, we get,

$$\begin{aligned} & M_1(M_2P_2 + P_2M_2^*) + (M_2P_2 + P_2M_2^*)M_1^* &= P_0 \\ \Rightarrow\; & M_2M_1P_2 + M_1P_2M_2^* + M_2P_2M_1^* + P_2M_1^*M_2^* &= P_0 \\ \Rightarrow\; & M_2(M_1P_2 + P_2M_1^*) + (M_1P_2 + P_2M_1^*)M_2^* &= P_0 \end{aligned}$$

Since M_2 is a Hurwitz matrix and P_0 is a positive definite matrix, we conclude from the last step that

$$M_1P_2 + P_2M_1^* < 0.$$

So P_2 is a common Lyapunov matrix for M_1 and M_2. It is also noticed that to compute P_2 we need to solve two Lyapunov equations (3.3) and (3.4).

Next we show that if a set of matrices $\{M_i\}$ are Hurwitz[1] and pairwise commutative then there exists a common Lyapunov matrix. Suppose given a positive definite matrix P_0, let $P_1, P_2, \ldots, P_l$ be the unique symmetric positive definite solutions to the Lyapunov equations

$$M_i P_i + P_i M_i^* = -P_{i-1}, i = 1, 2, \ldots, l. \tag{3.5}$$

Then we shall prove that the matrix P_l satisfies following Lyapunov equations

$$M_i P_l + P_l M_i^* < 0, i = 1, 2, \ldots, l. \tag{3.6}$$

Towards this end, define $P_{ij} = M_i P_j + P_j M_i^*$. If we can show that $P_{ij} < 0$ for $i = 1, 2, \ldots, l$, $j = i, i+1, \ldots, l$, then the result follows by choosing $j = l$ for each i. Hence, let $i \in \{1, 2, \ldots, l\}$. From (3.5), $P_{ii} = -P_{i-1} < 0$. Now assume that $P_{ij} < 0$ for some $j = \{1, 2, \ldots, l-1\}$. Then using (3.5) we find

$$\begin{aligned}
& M_{j+1} P_{i,j+1} + P_{i,j+1} M_{j+1}^* \\
= \;& M_{j+1}(M_i P_{j+1} + P_{j+1} M_i^*) + (M_i P_{j+1} + P_{j+1} M_i^*) M_{j+1}^* \\
= \;& M_i(M_{j+1} P_{j+1} + P_{j+1} M_{j+1}^*) + (M_{j+1} P_{j+1} + P_{j+1} M_{j+1}^*) M_i^* \\
= \;& -(M_i P_j + P_j M_i^*) \\
= \;& -P_{ij} \\
> \;& 0
\end{aligned}$$

This implies $P_{i,j+1} < 0$ and hence proves the results by induction.

Note that one has to solve l Lyapunov equations to find a common Lyapunov matrix P. Next we show that such an approach is not necessary and show a direct computation for a common Lyapunov matrix.

Lemma 3.2 *A finite set of Hurwitz diagonalizable commutative matrices $\{M_i\}$ share a common Lyapunov matrix and is given by $T^{-1}(T^{-1})^*$ where T is a nonsingular matrix which diagonalizes each matrix $\{M_i\}$.*

Proof : It is known from the matrix theory that diagonalizable[2] commutative matrices are simultaneously diagonalizable [30, 31]. Let T be the transformation matrix which

[1] 'A set of matrices $\{M_i\}$ are Hurwitz' means that every matrix of the given set $\{M_i\}$ is Hurwitz.

[2] Suppose matrix A_1 is diagonalizable by matrix T and $O_1 = T^{-1} A_1 T$. Then all nondiagonal elements of the matrix O_1 are zero

diagonalizes each matrix M_i, i.e.,

$$\delta_i = T^{-} \ M_i T^{-i}$$

where δ_i is the diagonal matrix whose diagonal elements correspond to the eigenvalues of matrix M_i. Define $P = T^{-1}(T^{-1})^*$. Using this matrix the following simplification is made.

$$\begin{aligned}
& M_i P + P M_i^* \\
= \ & T^{-1}\delta_i T P + P T^* \delta_i^* (T^{-1})^* \\
= \ & T^{-1} [\, \delta_i T P T^* + T P T^* \delta_i^* \,] (T^{-1})^* \\
= \ & T^{-1} [\, \delta_i + \delta_i^* \,] (T^{-1})^* \\
< \ & 0 \ \forall \, i
\end{aligned}$$

The last step is true by noticing that the the matrix $(\delta_i + \delta_i^*) < 0$ since the matrix M_i is Hurwitz. □

Remarks 3.1 : In the above lemma 3.2, we use the property of commutative matrices that they are simultaneously diagonalizable by matrix T. Here we shall discuss how to compute matrix T. For this purpose, we assume that a set of commutative matrices $\{A_1, A_2, \ldots, A_l\}$ is given and one of them is diagonalizable by matrix T For simplicity, suppose A_1 is diagonalizable by matrix T. This means that the matrix T is constructed by using the eigen vectors of matrix A_1 and $\delta_1 = T^{-1}A_1T$. Next we show that the matrix T diagonalizes all the matrices For this purpose, define $O_i = T^{-1}A_iT$ for $i = 1, 2, \ldots, l$. It is noted that the matrices $\{O_i\}$ are also commutative and the matrix $O_1 = \delta_1$ is a diagonal matrix.

$O_1 O_i = O_i O_1$ for $i = 2, 3, \ldots l$.

$\Rightarrow \delta_1 O_i = O_i \delta_1$ for $i = 2, 3, \ldots l$.

The above equation ensures that the matrices O_i are diagonal matrices. So the common Lyapunov matrix is given by $P = T^{-1}(T^{-1})^*$.

Next, we show the existence of a common Lyapunov matrix for a set of Hurwitz normal matrices.

Lemma 3.3 *If a set of matrices $\{M_i\}$ are Hurwitz and normal then there exists a common Lyapunov matrix P such that the following holds :*

$$M_i P + P M_i^* < 0 \ \forall \, i.$$

Proof : By assumptions, the matrices $\{M_i\}$ are normal, i.e., $M_iM_i^* = M_i^*M_i$ [30]. For these matrices, there exists orthonormal matrices T_i ($T_i^{-1} = T_i^*$) such that

$$T_i M_i T_i^* = \Delta_i.$$

Here the matrix Δ_i is the diagonal matrix whose diagonal elements contain the eigenvalues of matrix M_i since each matrix M_i is Hurwitz, $(\Delta_i + \Delta_i^*) < 0$.
Using $P = I_n$, the following is obtained

$$\begin{aligned} & M_iP + PM_i^* \\ = \ & T_i^*\Delta_iT_i + T_i^*\Delta_i^*T_i \\ = \ & T_i^*(\Delta_i + \Delta_i^*)T_i \\ < \ & 0 \end{aligned}$$

So the identity matrix is a common Lyapunov matrix for normal family. This completes the proof of the lemma. □

Next we introduce the following definition for partially commutative and normal control systems.

Definition 3.1 : A set of systems $\{A_i, B_i\}$ will be called partially commutative if the submatrices $\{A_{11}^i\}$, defined in (3.2), are Hurwitz and pairwise commutative, i.e.,

$$A_{11}^iA_{11}^j = A_{11}^jA_{11}^i \ \forall \ i, j.$$

Definition 3.2 : A set of systems $\{A_i, B_i\}$ will be called partially normal if the submatrices $\{A_{11}^i\}$, defined (3.2), are Hurwitz and normal, i.e.,

$$A_{11}^i(A_{11}^i)^* = (A_{11}^i)^*A_{11}^i \ \forall \ i.$$

Based on the above definition, we shall state the following results for SQ stabilization.

Theorem 3.1 *A finite set of systems $\{A_i, B_i\}, i = 1, 2, \ldots, l$ of the form (3.1) are simultaneously quadratically stabilizable if the matrices $\{A_{11}^i\}$ are Hurwitz and commutative.*

Proof : From theorem 2.6, the set of systems $\{A_i, B_i\}$ are SQ stabilizable iff there exists matrices F and W_1 such that the following holds

$$\left[\left(A_{11}^i + A_{12}^iF\right) W_1 + W_1 \left(A_{11}^i + A_{12}^iF\right)^* \right] < 0 \ \forall \ i$$

By hypotheses, the matrices $\{A^i_{11}\}$ are Hurwitz and pairwise commutative. So choosing $F = \mathbf{0}$, the above inequalities are reduced to

$$A^i_{11}W_1 + W_1(A^i_{11})^* < 0 \,\forall\, i$$

But using lemma 3.1 or lemma 3.2, there exists a single matrix W_1, such that the above inequality holds for all i. The construction of such a W_1 matrix is discussed in the proof of lemmas 3.1-3.2. Using these matrices F $(= \mathbf{0})$ and W_1 in the design algorithm (described in section 2.9, page 33), one can compute state feedback controller □

Example 3.1 : Consider a set of three systems $\{A_i, B\}$ described in (3.1) where

$$A_1 = \begin{bmatrix} -1 & 1 & 0 \\ 0.5 & -2 & 1 \\ 1 & 0 & 1 \end{bmatrix}, = \begin{bmatrix} A^1_{11} & A^1_{12} \\ A^1_{21} & A^1_{22} \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -0.5 & 1 & 0.51 \\ 0.5 & -1.5 & 1 \\ 1 & 0 & 1.3 \end{bmatrix} = \begin{bmatrix} A^2_{11} & A^2_{12} \\ A^2_{21} & A^2_{22} \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -1.5 & 1 & 0 \\ 0.5 & -2.5 & -0.1 \\ 0 & 1 & 0.5 \end{bmatrix} = \begin{bmatrix} A^3_{11} & A^3_{12} \\ A^3_{21} & A^3_{22} \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

It is noticed from the above that the submatrices $\{A^i_{11}\}$ are not same for all the systems. So these three systems do not belong to the matched family of systems. However the matrices $\{A^i_{11}\}, i = 1, 2, 3$ are Hurwitz and commutative. Hence these systems are stabilizable by a single controller.

Using controller algorithm (described in section 2.9), we have the Lyapunov matrix

$$W = \begin{bmatrix} 0.9553 & 0.2277 & 0.0 \\ 0.2277 & 0.7053 & 0.0 \\ 0.0 & 0.0 & 2.0 \end{bmatrix},$$ and controller gain is given by $K = \begin{bmatrix} 0.0 & 0.0 & 25 \end{bmatrix}$.

Next we show the existence of a stabilizing controller for a family of partially normal systems.

Theorem 3.2 *A finite set of partially normal systems $\{A_i, B_i\}, i = 1, 2, \ldots, l$ of the form (3.1) is simultaneously quadratically stabilizable.*

Proof : From theorem 2.6, the set of system $\{A_i, B_i\}$ is SQ stabilizable iff there exists matrices F and W_1 such that the following holds

$$\left[\left(A_{11}^i + A_{12}^i F\right) W_1 + W_1 \left(A_{11}^i + A_{12}^i F\right)^* \right] < 0 \;\forall\; i$$

By hypotheses, the matrices $\{A_{11}^i\}$ are Hurwitz and normal. So choosing $F = \mathbf{0}$, the above inequalities are reduced to

$$\left[A_{11}^i W_1 + W_1 (A_{11}^i)^* \right] < 0 \;\forall\; i$$

But using lemma 3.3, there exists a single matrix $W_1 = I_{n-m}$, such that the above inequalities hold for all $i = 1, 2, \ldots, l$. Using these matrices F and W_1 in the design algorithm (described section 2.9), one can compute state feedback controller. This completes the proof. □

Example 3.2 : Consider a set of three systems $\{A_i, B\}$ described in (3.1) where

$$A_1 = \begin{bmatrix} -2 & -1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -2 & 1 \\ 0.5 & -1 & 1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 1 & 1 & 0.5 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

Using controller algorithm (described in section 2.9), we have the Lyapunov matrix $W = \begin{bmatrix} 1 & 0.0 & 0.0 \\ 0.0 & 1 & 0.0 \\ 0.0 & 0.0 & 2.0 \end{bmatrix}$, and controller gain is given by $K = \begin{bmatrix} 0.0 & 0.0 & 5 \end{bmatrix}$. It is noticed that these three systems are different from matched uncertain systems.

3.2 SQ Stabilization for More General Class of Systems

In this section, we shall derive results for some general class of systems. For the set of systems $\{A_i, B_i\}$ the following assumptions are made

Assumption 3.1 : The number of inputs $m \geq n/2$ if n is even and $m \geq (n+1)/2$ if n is odd.

Assumption 3.2 : Each submatrix $A_{12}^i \in R^{n-m \times m}$ is a full rank matrix, i.e., rank $(A_{12}^i) = n - m$.

Definition 3.3 : A set of full rank matrices $\{A_{12}^i\}$ of the partition (3.2) will be called as quasi sign invariant matrices if for any matrix $E \in R^{m \times n-m}$ the following conditions hold

$$(A_{12}^i E + (A_{12}^i E)^*) > 0 \; \forall \, i.$$

As an example, suppose all the matrices A_{12}^i are constant and $A_{12}^i = A_{12}$ for all i (say). In this case, the matrix E can be taken as $E = A_{12}^*$.

Theorem 3.3 *A finite set of systems $\{A_i, B_i\}, i = 1, 2, \ldots, l$ is simultaneously quadratically stabilizable if the matrices $\{A_{12}^i\}$ are quasi sign invariant matrices.*

Proof : For the stabilizability of the given set of systems $\{A_i, B_i\}$, the following inequalities hold from theorem 2.6

$$\left[\left(A_{11}^i + A_{12}^i F\right) W_1 + W_1 \left(A_{11}^i + A_{12}^i F\right)^* \right] < 0 \; \forall \, i. \tag{3.7}$$

Since the matrices $\{A_{12}^i\}$ are quasi sign invariant, so by definition 3.3, there exists a matrix E such that

$$(A_{12}^i E + (A_{12}^i E)^*) > 0 \; \forall \, i.$$

Using $W_1 = I_{n-m}$ and $F = -\alpha E$ where $\alpha > 0$, the inequalities (3.7) can be written as

$$A_{11}^i + (A_{11}^i)^* - \alpha(U_i + U_i^*) < 0 \; \forall \, i.$$

where $U_i = A_{12}^i E$. Since the matrix $(U_i + U_i^*) > 0$, the inequalities (3.7) will be satisfied by taking sufficiently large scalar α. The value of such a α can be selected as follows : For each index i, define $\alpha_i = \lambda_M[(U_i + U_i^*)^{-1}(A_{11}^i + (A_{11}^i)^*)]$ and

$$\alpha = \max\{\alpha_i\}.$$

Using this $W_1 (= I_{n-m})$ and $F = -\alpha E$ in the main algorithm of chapter 2 (section 2.9, page 33), one can compute controller gain. □

3.3 Stabilization of a Continuously Parameterized Family of Systems

In the previous sections, we have considered the SQ stabilization of finite number LTI systems described by

$$\dot{x} = A_i x + B_i u, \quad i = 1, 2, \ldots, l.$$

Here each of the above systems (A_i, B_i) is referred as an extreme system. Now suppose a finite set of extreme system $\{A_i, B_i\}$ is SQ stabilizable. Then the question is asked whether a set of systems (constructed by the positive linear combination of the given extreme plants) is also SQ stabilizable.

For this purpose, define a parameterized family of systems described by

$$\dot{x} = A(\alpha)x + B(\alpha)u \tag{3.8}$$

where $A(\alpha) = \sum_{i=1}^{l} \alpha_i A_i$ and $B(\alpha) = \sum_{i=1}^{l} \alpha_i B_i$, $\alpha_i \geq 0 \; \forall \, i$, and $\sum_{i=1}^{l} \alpha_i > 0$.
Here each matrix B_i satisfies the assumption 2.3 (page 32), i.e.,

$$B_i = BG_i \text{ where } B = \begin{bmatrix} 0 \\ I_m \end{bmatrix} \text{ and } (G_i + G_i^*) > 0.$$

Next we prove that this parameterized family is also SQ stabilizable.

Theorem 3.4 *The parametrized family (A_α, B_α) described in (3.8) is simultaneously quadratically stabilizable by a single state feedback controller if any one of the following holds*

(a) $\{A_i, B_i\}$ are matched systems

or

(b) $\{A_i, B_i\}$ are partially commutative or partially normal

or

(c) $\{A_i, B_i\}$ are in Hessenberg family.

Proof : The proof of this theorem is done by employing the results of chapter 2 and 3. Here we are providing the proof of part (a) and the other proofs will be similar.

proof of (a) : Since the systems $\{A_i, B_i\}$ satisfy the matching condition, there exists a controller gain K and matrix $W > 0$ such that the following inequalities holds by theorems 2.7 and 2.3

$$(A_i - B_i K)^* W^{-1} + W^{-1}(A_i - B_i K)^* < 0 \; \forall \, i = 1, 2, \ldots, l. \tag{3.9}$$

Next we shall show that the following inequality holds true.

$$(A_\alpha - B_\alpha K)^* W^{-1} + W^{-1}(A_\alpha - B_\alpha K) < 0$$

Now we shall simplify the above using (3.9) in the following way.

$$\begin{aligned} & (A_\alpha - B_\alpha K)^* W^{-1} + W^{-1}(A_\alpha - B_\alpha K) \\ = & \sum_{i=1}^{l} \alpha_i [(A_i - B_i K)^* W^{-1} + W^{-1}(A_i - B_i K)^*] \\ < & \ 0. \end{aligned}$$

The third step is true due to (3.9) and noting the fact that positively weighted sum of negative definite matrices are negative definite. This ensures that the closed loop matrix $(A_\alpha - B_\alpha K)$ is Hurwitz using lemma 2.1. □.

3.4 Conclusion

In this chapter, we solve the SQ stabilization problem of partial commutative family and partial normal family of LTI systems. The existence conditions are ensured for these class of systems. The results for some more general class of systems are also derived. It is important to note that these classes of systems are different from that of matched uncertain systems. Hypothetical numerical examples are also provided to verify these results.

Chapter 4

Simultaneous Stabilization by Output Feedback

In this chapter, new classes of LTI systems are identified for the simultaneous quadratic stabilization problem by static output feedback.

4.1 Introduction

Design of static state feedback controller for the control system design has been discussed in the previous two chapters. In these cases the feedback controller of the form $u = -Kx(t)$ has been used. This type of controller requires knowledge of only instantaneous value of state $x(t)$ and hence one must have all the state variables $x(t)$ available for feedback. However, in practice, it is not always possible to measure all the states. So the design of static output feedback controller by using measurable states is a more realistic approach in control theory.

Recently a sufficient condition for output feedback stabilizability of linear matched uncertain systems has been developed in [65, 64, 83, 27]. It has been established that if the nominal system is minimum phase and has relative degree of unity, then robust stabilization is possible using static output feedback controller [65, 83]. In [27], this result is extended to multivariable systems with equal number of inputs and outputs. In this chapter, we report new classes of systems which will be SQ stabilizable by static output feedback.

Consider a family of multi-input multi-output (MIMO) systems $\{A_i, B_i, C\}$ $i = 1, 2, \ldots, l$

described by the state equations,

$$\dot{x} = A_\imath x + B_\imath u, y = Cx \tag{4.1}$$

where $x \in R^n$ is the state vector, $u \in R^m$ denotes control input and $y \in R^p$ is measured output. To stabilize the above set of systems, the following output feedback controller is considered

$$u = -Ly(t). \tag{4.2}$$

The basic problem is to design a matrix L such that the closed loop matrix for each system $(A_\imath - B_\imath LC)$ is Hurwitz, i.e., all the eigenvalues of $(A_\imath - B_\imath LC)$ are in the open left half of the complex plane.

Before going to the actual discussion, here we show how dynamic output feedback can conceptually be replaced by static output feedback [7]. More specifically, suppose we want to design a v^{th} order dynamic output feedback controller of the form

$$\dot{x}_c = A_c x_c + B_c y, u = C_c x_c + D_c y. \tag{4.3}$$

The augmented system (using 4.1 and 4.3) can be written as

$$\dot{x_a} = A_{a\imath} x_a + B_{a\imath} u_a, \quad y_a = C_a x_a, \quad u_a = \bar{K} y_a \tag{4.4}$$

where $x_a = [x^* \ x_c^*]^*$, $u_a = [u^* \ \dot{x}_c^*]^*$, $y_a = [y^* \ x_c^*]^*$,

$$A_{a\imath} = \begin{bmatrix} A_\imath & 0 \\ 0 & 0 \end{bmatrix}, B_{a\imath} = \begin{bmatrix} B_\imath & 0 \\ 0 & I_v \end{bmatrix}, C_a = \begin{bmatrix} C & 0 \\ 0 & I_v \end{bmatrix}, \bar{K} = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}.$$

So the design of dynamic output feedback controller is equivalent to the design of static output feedback controller for the augmented system.

4.2 Problem Definition and Results

In general the design of output feedback controller is difficult both from the existence condition as well as algorithmic point of view. So for the design of matrix $L \in R^{m \times m}$, we have combined the Lyapunov stability approach and some mathematical tools for the solution of matrix inequalities.

Definition 4.1 : A set of systems (4.1) is simultaneously quadratically stabilizable by static output feedback if there exists a matrix W and matrix L such that the following inequalities holds

$$(A_i - B_i LC)^* W^{-1} + W^{-1}(A_i - B_i LC) < 0 \, \forall \, i. \; \square$$

For the set of systems (4.1), following assumptions are made.

Assumption 4.1 : The matrix $B_i = BG_i$ where nonsingular matrix $G_i \in R^{m \times m}$ is such that $(G_i + G_i^*) > 0$.

Assumption 4.2 : The number of control inputs is equal to the number of measurable outputs, i.e., $p = m$. The matrix $B = \begin{bmatrix} 0 \\ I_m \end{bmatrix}$ and $C = [C_1 \; C_2]$ where $C_2 \in R^{m \times m}$ is a nonsingular matrix. These particular structures of B_i and C matrices ensure that the determinant of the matrix $[CB_i]$ is not equal to zero, i.e., $det[CB_i] \neq 0$.

Assumption 4.3 : The transfer function matrix $C(sI_n - A_i)^{-1}B_i$ for each system is strictly minimum phase i.e., all the zeros of the system will lie in the open left half of complex plane.

In the above assumption 4.1, the particular structure of B matrix is always possible using a similarity transformation. Notice that these assumptions (4.2 -4.3) are made for the sake of convenience. The expression $det[CB_i]$ is known as the high frequency gain of the system (A_i, B_i, C) [27].

To design output feedback controller, we consider the state feedback controller described in chapter 2. In chapter 2, the state feedback controller, of the form $u = -Kx(t) = -\gamma B^* W^{-1} x(t)$ where γ is a positive scalar, is used. The W matrix is the positive definite solution of the following matrix inequalities (from lemma 2.4)

$$A_i W + W A_i^* - (\gamma/2)(B_i B^* + B B_i^*) < 0 \, \forall \, i \tag{4.5}$$

In the above (Riccati) inequalities, γ is a free parameter which must be chosen properly to ensure the positive definiteness of W. When all the states are not available, one needs to consider the output feedback controller. Due to the formulation (4.4) above we look for ways of designing the output feedback from the state feedback law. In view of (4.5)

and the above discussions, we restate the problem of SQ stabilization by output feedback controller as follows :

Definition 4.2 : A set of systems in (4.1) is simultaneously quadratically stabilizable by static output feedback if there exists a positive definite matrix W and a matrix L such that the following conditions hold for some positive scalar γ :

$$(a) \quad A_iW + WA_i^* - (\gamma/2)(B_iB^* + BB_i^*) < 0 \; \forall \; i \tag{4.6}$$

$$(b) \quad LC = (\gamma/2)B^*W^{-1}. \tag{4.7}$$

In the above definition, the state feedback controller $K = (\gamma/2)B^*W^{-1}$ guarantees the stability due to (4.6), while due to (4.7) there is an output feedback gain L corresponding to state feedback gain K. Note that if (4.6-4.7) hold for each $i = 1, 2, \ldots, l$ then $V(x) = x^*W^{-1}x$ will be a quadratic Lyapunov function for each closed loop system.
Let the matrices A_i, B_i, C be partitioned as follows :

$$A_i = \begin{bmatrix} A_{11}^i & A_{12}^i \\ A_{21}^i & A_{22}^i \end{bmatrix}, \; B_i = BG_i = \begin{bmatrix} 0 \\ I_m \end{bmatrix} G_i, \; C = [C_1 \; C_2] \tag{4.8}$$

where $A_{11}^i \in R^{n-m \times n-m}$, $A_{12}^i \in R^{n-m \times m}$ and G_i is a nonsingular matrix.

Theorem 4.1 *Suppose the system (4.1) satisfies the assumptions 4.1-4.3 and matrices A_i, B_i, C are partitioned as given in (4.8). A set of systems $\{A_i, B_i, C\}$ are simultaneously quadratically stabilizable by a output feedback control law $u = -Ly$ with a nonsingular matrix L iff there exists a positive definite solution W_1 of the following inequalities*

$$(A_{11}^i - A_{12}^iF)W_1 + W_1(A_{11}^i - A_{12}^iF)^* < 0 \; \forall \; i \tag{4.9}$$

where $F = C_2^{-1}C_1$.

Proof : (Sufficiency) :
By hypothesis, there exists a matrix $W_1 = W_1^* > 0$ such that the following inequalities hold,

$$[(A_{11}^i - A_{12}^iF)W_1 + W_1(A_{11}^i - A_{12}^iF)^*] < 0 \; \forall \; i.$$

We shall prove that there exists a matrix W and a nonsingular matrix L such that inequalities (4.6-4.7) hold. For this purpose, we construct a positive definite matrix W in the following way. With the help of matrices F and W_1, we choose

$$W_2^* = -FW_1. \tag{4.10}$$

Then, we select W_3 such that $W_3 - W_2^* W_1^{-1} W_2 > 0$ which ensures that $W = \begin{bmatrix} W_1 & W_2 \\ W_2^* & W_3 \end{bmatrix}$ is a positive definite matrix. Such a choice of W_3 is always possible as shown in the proof of theorem 2.1 of chapter 2.

Using this matrix W, consider the following matrices

$$\Phi_i = A_i W + W A_i^* - (\gamma/2)(B_i B^* + B B_i^*). \tag{4.11}$$

Now the problem in hand is to show that $\{\Phi_i\}$, $i = 1, 2, \ldots, l$, are negative definite and there exists a nonsingular matrix L such that (4.7) is satisfied.

Now from (4.10) the matrix $F = -W_2^* W_1^{-1}$ while by hypothesis of the theorem matrix $F = C_2^{-1} C_1$. Hence (4.10) implies :

$$W_2^* = -C_2^{-1} C_1 W_1 \Rightarrow C_1 W_1 + C_2 W_2^* = 0. \tag{4.12}$$

We partition $\Phi_i = \begin{bmatrix} \phi_{11}^i & \phi_{12}^i \\ (\phi_{12}^i)^* & \phi_{22}^i \end{bmatrix}$ with $\phi_{11}^i \in R^{n-m \times n-m}$. Then, after simplifying (4.11) and using $F = -W_2^* W_1^{-1} = C_2^{-1} C_1$, we get

$$\phi_{11}^i = (A_{11}^i - A_{12}^i F) W_1 + W_1 (A_{11}^i - A_{12}^i F)^* \tag{4.13}$$

$$\phi_{22}^i = (A_{21}^i W_2 + A_{22}^i W_3) + (A_{21}^i W_2 + A_{22}^i W_3)^* - (\gamma/2) I_m (G_i + G_i^*) \tag{4.14}$$

$$\phi_{12}^i = A_{11}^i W_2 + A_{12}^i W_3 + W_1 (A_{21}^i)^* + W_2 (A_{22}^i)^* \tag{4.15}$$

The matrix Φ_i will be negative definite if [38]

$$(I)\ \phi_{11}^i < 0 \text{ and } (II)\ \phi_{22}^i - (\phi_{12}^i)^* \, (\phi_{11}^i)^{-1} \, \phi_{12}^i < 0.$$

Observe that the matrix ϕ_{11}^i is negative definite for each $i = 1, 2, \ldots, l$ by the hypothesis of theorem. Also observe that ϕ_{11}^i and ϕ_{12}^i are independent of γ and the matrix $(G_i + G_i^*) > 0$. So the matrix ϕ_{22}^i can be made negative definite by choosing a sufficiently large value of γ. Since ϕ_{11}^i and ϕ_{12}^i are independent of γ, we can choose γ such that

$$\phi_{22}^i - (\phi_{12}^i)^* \, (\phi_{11}^i)^{-1} \, \phi_{12}^i < 0 \; \forall \, i.$$

The value of γ is computed as follows :
Using the expression of ϕ_{22}^i from (4.14), the above inequalities are rewritten as

$$(\gamma/2)[G_i + G_i^*] > [S_i - (\phi_{12}^i)^* (\phi_{11}^i)^{-1} \phi_{12}^i] \; \forall \, i \tag{4.16}$$

where matrix $S_i = (A_{21}^i W_2 + A_{22}^i W_3) + (A_{21}^i W_2 + A_{22}^i W_3)^*$.
Now for each index i, choose a scalar γ_i such that

$$\gamma_i > 2\lambda_M \left[(S_i - (\phi_{12}^i)^* (\phi_{11}^i)^{-1} \phi_{12}^i)[G_i + G_i^*]^{-1} \right].$$

Let the value of $\gamma = \max\{\gamma_i\}, i = 1, 2, \ldots, l$. For this value of γ and W, the matrix Φ_i, defined in (4.11), will be negative definite for all i.

The existence of nonsingular matrix L is discussed as follows.
The matrix W is a positive definite matrix. So $CWC^* > 0$. By partitioning C and W we write

$$\begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} W_1 & W_2 \\ W_2^* & W_3 \end{bmatrix} \begin{bmatrix} C_1^* \\ C_2^* \end{bmatrix} > 0.$$

Using (4.12), the above inequality is further simplified and written as

$$\begin{bmatrix} C_1 W_2 + C_2 W_3 \end{bmatrix} C_2^* > 0.$$

Since the matrix C_2 is a nonsingular, the above inequality ensures that the matrix $[\, C_1 W_2 + C_2 W_3 \,]$ is a nonsingular matrix.
Let us define the matrix

$$L = (\gamma/2)[\, C_1 W_2 + C_2 W_3 \,]^{-1}. \tag{4.17}$$

Using this matrix L, we observe that (using (4.12))

$$\begin{aligned} LCW^* &= (\gamma/2)[\, C_1 W_1 + C_2 W_3 \,]^{-1} \, [C_1 W_1 + C_2 W_2^* \quad C_1 W_2 + C_2 W_3] \\ &= (\gamma/2) B^*. \end{aligned}$$

Thus using the matrix L in (4.17), equation (4.7) is satisfied. Hence the controller $u = -Ly(t)$ stabilizes the family of systems.

Necessary : Suppose there exists a W and a nonsingular matrix L such that (4.6) and (4.7) hold. Consider the matrix

$$\Phi_i = A_i W + W A_i^* - (\gamma/2)(B_i B^* + B B_i^*) < 0. \tag{4.18}$$

Now from (4.7) (note that $B = \begin{bmatrix} 0 \\ I_m \end{bmatrix}$ and L is a nonsingular matrix), we write

$$CW = (\gamma/2)L^{-1}B^* \Rightarrow C_1W_1 + C_2W_2^* = 0,\ C_1W_2 + C_2W_3 = (\gamma/2)L^{-1} \tag{4.19}$$

Using (4.19), we partitioned the matrix $\Phi_i = \begin{bmatrix} \phi_{11}^i & \phi_{12}^i \\ (\phi_{12}^i)^* & \phi_{22}^i \end{bmatrix}$ of (4.18) where the submatrix ϕ_{11}^i is given by

$$\phi_{11}^i = (A_{11}^i - A_{12}^i F)W_1 + W_1(A_{11}^i - A_{12}^i F)^*$$

Since the matrices $\{\Phi_i\}$ are negative definite, so are the matrices $\{\phi_{11}^i\}$. Hence the proof. □

Remarks 4.1 : We discussed the static output feedback stabilizability of square systems (equal number of inputs and outputs). These results can be applied to nonsquare systems by squaring down the systems. The results of squaring down of nonsquare LTI system reported in [58] may be useful for this purpose.

To verify the existence conditions given in (4.9) of theorem 4.1, one needs to solve the following Lyapunov equations for the existence of a single matrix W_1 :

$$\left(A_{11}^i - A_{12}^i F\right) W_1 + W_1 \left(A_{11}^i - A_{12}^i F\right)^* < 0 \ \forall\, i.$$

However, the necessary condition for the existence of a common solution of the above inequalities is that the matrices $\{A_{11}^i - A_{12}^i F\}$ must be Hurwitz for each i. Due to the above requirement, we shall state a lemma which will be useful to prove the existence conditions given in (4.9). In the following development, the notion of system zeroes [56] are used to define minimum phase systems. A system will be called minimum phase system if the zeroes of the given system lies in the open left half of complex plane.

Lemma 4.1 *Suppose that the matrices A_i, B_i, C are partitioned as in (4.8). Then the transfer function $C(sI_n - A_i)^{-1}B_i$ is minimum phase iff the matrix $(A_{11}^i - A_{12}^i C_2^{-1} C_1)$ is Hurwitz, i.e., all the eigen-values of matrix $(A_{11}^i - A_{12}^i C_2^{-1} C_1)$ have negative real parts.*

Proof : This result is based on [26] and proved as follows.

From the definition of zeros [56], rank $\begin{bmatrix} sI_{n-m} - A_{11}^i & -A_{12}^i & 0 \\ -A_{21}^i & sI_m - A_{22}^i & G_i \\ -C_1 & -C_2 & 0 \end{bmatrix}$ will decrease for

any value of s which is a zero of the system realization $\{A_i, B_i, C\}$. It is also known that multiplication of a nonsingular matrix does not change the rank of a matrix.

Define the matrix

$$Z = \begin{bmatrix} sI_{n-m} - A^i_{11} & -A^i_{12} & 0 \\ -A^i_{21} & sI_m - A^i_{22} & G_i \\ -C_1 & -C_2 & 0 \end{bmatrix} \begin{bmatrix} I_{n-m} & 0 & 0 \\ -C_2^{-1}C_1 & I_m & 0 \\ 0 & 0 & G_i^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} sI_{n-m} - A^i_{11} + A^i_{12}C_2^{-1}C_1 & \star & 0 \\ \star & \star & I_m \\ 0 & -C_2 & 0 \end{bmatrix}$$ where the submatrices denoted by $\star$ can assume any form. The rank of above matrix will decrease if s is an eigenvalue of matrix $(A^i_{11} - A^i_{12}C_2^{-1}C_1)$. □

In theorem 4.1, we derive existence conditions for the SQ stabilization of a set of system $\{A_i, B_i, C\}$ by a static output feedback controller. These conditions are derived in terms of matrix inequalities defined in (4.9). But what are the structures of $\{A_i, B_i, C\}$ such that SQ stabilization problem is solvable by output feedback? The rest of this chapter is devoted to obtain such classes of systems.

4.3 Stabilization of Matched Systems by Output Feedback

The concept of matching condition for linear time varying uncertain system is discussed in chapter 1 (page 5). Using matching condition[1], robust stabilization by output feedback is solved in [27]. However, in the present context, we shall call a set of LTI system $\{A_i, B_i, C\}$ as matched system if the submatrices $\{A^i_{11}\}$ and $\{A^i_{12}\}$ are same for all the systems. The formal definition is as follows.

Definition 4.3 : A set of LTI systems $\{A_i, B_i, C\}$ will be called matched system if the submatrices $\{A^i_{11}\}$ and $\{A^i_{12}\}$ are same for all the systems.
Defining this, the SQ stabilization results of such systems is stated as follows.

[1] Consider an uncertain systems $\dot{x} = (A_0 + \Delta A)x + (B_0 + \Delta B)u, y = Cx$. This system is said to satisfy matching condition [27] if $\Delta A = B_0 O_1$ and $\Delta B = B_0 O_2$ where $\|O_2\| < 1$.

Theorem 4.2 *A finite set of matched systems $\{A_i, B_i, C\}, i = 1, 2, \ldots, l$ described in (4.1) is simultaneously quadratically stabilizable by static output feedback controller.*

Proof : To prove the above theorem consider the special structure of A_i and B_i matrices as partitioned in (4.8). By definition 4.1, the matrices A_i and B_i defined for matched system are as follows

$$A_i = \begin{bmatrix} A_{11} & A_{12} \\ A_{21}^i & A_{22}^i \end{bmatrix} \text{ and } B_i = \begin{bmatrix} 0 \\ I_m \end{bmatrix} G_i.$$

For the stabilizability of the given set of systems, the following expression will be negative definite, i.e.,

$$[\,(A_{11} - A_{12}F)\,W_1 + W_1\,(A_{11} - A_{12}F)^*\,] < 0 \tag{4.20}$$

where $F = C_2^{-1}C_1$.

This is a Lyapunov inequality. Since $\{A_i, B_i, C\}$ are minimum phase, the matrix $(A_{11} - A_{12}F)$ is Hurwitz by lemma 4.1. Hence, there exists a matrix $W_1 > 0$ such that (4.20) holds. The matrix W_1 is computed by solving the following Lyapunov equation

$$[\,(A_{11} - A_{12}F)\,W_1 + W_1\,(A_{11} - A_{12}F)^*\,] = Q_0$$

where Q_0 is any negative definite matrix of dimension $(n - m \times n - m)$. The matrices W_1 and F are used in the step 1 of the algorithm (described in section 4.5, page 57) to find controller gain. □

As a special case of theorem 4.3, it is proved in next theorem that if each matrix $A_i, i = 1, 2, \ldots, l$ is in block companion form [62], then the systems $\{A_i, B_i, C\}$ are SQ stabilizable.

Theorem 4.3 *A finite set of systems $\{A_i, B_i, C\}, i = 1, 2, \ldots, l$ in block controllable companion form is simultaneously quadratically stabilizable by a static output feedback.*

Proof : To prove the above theorem we consider the special structure of A_i and B_i matrices of (4.1) as given below,

$$A_i = \begin{bmatrix} 0 & I_m & 0 & \ldots & 0 \\ 0 & 0 & I_m & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & I_m \\ -\alpha_0^i & -\alpha_1^i & -\alpha_2^i & \ldots & -\alpha_r^i \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21}^i & A_{22}^i \end{bmatrix} \tag{4.21}$$

and $B_i = \begin{bmatrix} 0 \\ I_m \end{bmatrix} G_i$ where $r = n/m$.

Note that for each system the matrices A_{11} and A_{12} are same. Using theorem 4.2, we can conclude that $\{A_i, B_i, C\}$ are SQ stabilizable by an output feedback controller. This completes the proof of the theorem. □

4.4 Results for Mismatched Systems

In the previous section, we discussed the stabilization of the matched systems by output feedback controller. In the present section, we identify two new classes of systems which are stabilizable by single output feedback controller. These classes of systems are different from matched systems. Towards this end, we shall prove the following results for the SQ stabilization problem by output feedback.

Theorem 4.4 *A finite set of LTI systems $\{A_i, B_i, C\}$, as partitioned in (4.8), is simultaneously quadratically stabilizable by static output feedback if the matrices $\{A_{11}^i - A_{12}^i C_2^{-1} C_1\}$ are Hurwitz and commutative.*

Proof : From theorem 4.1, the set of system $\{A_i, B_i, C\}$ is SQ stabilizable iff there exists a positive definite matrix W_1 such that the following holds

$$\left[\left(A_{11}^i - A_{12}^i F\right) W_1 + W_1 \left(A_{11}^i - A_{12}^i F\right)^* \right] < 0 \; \forall \, i \tag{4.22}$$

By the hypotheses of the theorem, the matrices $\{A_{11}^i - A_{12}^i F\}$ are Hurwitz and pairwise commutative. Hence using lemma 3.1 of chapter 3, there exists a single matrix W_1, such that (4.22) holds for all i. This matrix W_1 is used in the design algorithm (described in section 4.5, page 57), to compute output feedback controller. □

Theorem 4.5 *A finite set of LTI systems $\{A_i, B_i, C\}$, as partitioned in (4.8), is simultaneously quadratically stabilizable by static output feedback if the matrices $\{A_{11}^i - A_{12}^i C_2^{-1} C_1\}$ are Hurwitz and normal.*

Proof : From theorem 4.1, the set of system $\{A_i, B_i, C\}$ is SQ stabilizable by a static output feedback controller iff there exists a matrix W_1 such that the following holds

$$\left[\left(A_{11}^i - A_{12}^i C_2^{-1} C_1\right) W_1 + W_1 \left(A_{11}^i - A_{12}^i C_2^{-1} C_1\right)^* \right] < 0 \; \forall \, i \tag{4.23}$$

By the hypotheses of the theorem, matrices $\{A_{11}^i - A_{12}^i C_2^{-1} C_1\}$ are Hurwitz and normal. Hence using lemma 3.3 of chapter 3, there exists a single matrix $W_1 = I_{n-m}$, such that the (4 23) holds for all i. Controller gain is computed using this matrix W_1 in the design algorithm (described in section 4 5, page 57). This completes the proof □

Remarks 4.2 : It is observed from definition 4.1 that a set of systems $\{A_i, B_i, C\}$ are matched if the submatrices $\{A_{11}^i\}$ and $\{A_{12}^i\}$ are same for all the systems. However, in theorem 4 3 or theorem 4.4, we consider a class of systems for which the submatrices $\{A_{11}^i\}$ and $\{A_{12}^i\}$ are not needed to be same for all the systems. In this sense, they are different from the matched systems. This will be also clear from the example 4.1.

4.5 Controller Design Algorithm

Based on the above proof of theorem 4.1, we present successive steps to compute a static output feedback controller gain for the finite set of systems $\{A_i, B_i, C\}, i = 1, 2, \ldots, l$ defined in (4.1). In this case it is assumed that all the systems are SQ stabilizable by an output feedback controller. So from theorem 4.1, there exists a positive definite matrix W_1 such that

$$\left(A_{11}^i - A_{12}^i F\right) W_1 + W_1 \left(A_{11}^i - A_{12}^i F\right)^* < 0 \;\forall\, i$$

where $F = C_2^{-1} C_1$.

Step 0 : This step is for the computation of W_1 such that above inequalities hold. In the previous sections of this chapter, existence conditions (hence the construction of W_1 matrix) are discussed for the following cases :

- Matched systems
- Matrices $\{A_{11}^i - A_{12}^i F\}$ are Hurwitz and pairwise commutative
- Matrices $\{A_{11}^i - A_{12}^i F\}$ are Hurwitz and normal.

The W_1 is subsequently used for the controller design purpose.

Step-1 : Using matrices W_1 and F, compute Q_0^i such that $Q_0^i = (A_{11}^i - A_{12}^i F)\, W_1 + W_1\, (A_{11}^i - A_{12}^i F)^*$ for each i. Here Q_0^i is $(n-m \times n-m)$ dimensional negative definite matrix

Step-2 : Compute $W_2^* = -FW_1$

Step-3 : Select W_3 such that $W_3 > W_2^* W_1^{-1} W_2$ to ensure $W = \begin{bmatrix} W_1 & W_2 \\ W_2^* & W_3 \end{bmatrix}$ be a positive definite matrix. Choose $W_3 = R + W_2^* W_1^{-1} W_2$ where R is a positive definite matrix.

Step-4 : Using (4.6) and (4.7), compute the value of γ, and the required controller gain is given by $L = (\gamma/2)(C_1 W_2 + C_2 W_3)^{-1}$. The computation of γ to satisfy (4.6-4.7) is simplified as follows.

Define $\Phi_i = A_i W + W A_i^* - (\gamma/2)(B_i B^* + B B_i^*) = \begin{bmatrix} Q_0^i & q_i \\ q_i^* & q_{2i} - (\gamma/2)(G_i + G_i^*) \end{bmatrix}$ where

$q_i = A_{11}^i W_2 + A_{12}^i W_3 + W_1 (A_{21}^i)^* + W_2 (A_{22}^i)^*$ and
$q_{2i} = A_{21}^i W_2 + A_{22}^i W_3 + W_2^* (A_{21}^i)^* + W_3 (A_{22}^i)^*$.

Now the problem is to find γ such that the matrices $\{\Phi_i\}$ will be negative for each $i = 1, 2, \ldots, l$. Note that, the matrix Q_0^i is a negative definite matrix defined in Step-1. So the matrices $\{\Phi_i\}$ will be negative definite if [38]

$$(i)\ Q_0^i < 0 \quad (ii)\ q_{2i} - (\gamma/2)(G_i + G_i^*) - q_i^* (Q_0^i)^{-1} q_i < 0\ \forall\, i.$$

Now, For each index i, compute γ_i such that the following holds :

$$\gamma_i > 2\lambda_M[\, [\, q_{2i} - q_i^* (Q_0^i)^{-1} q_i\,]\, [\, G_i + G_i^*\,]^{-1}\,].$$

The required γ is selected as the maximum of all such scalars $\{\gamma_i\}$, i.e.,

$$\gamma = \max\{\gamma_i\}.$$

This completes the design algorithm □

Example 4.1 : Consider a set of three systems $\{A_i, B_i, C\}, i = 1, 2, 3$ where

$$A_1 = \begin{bmatrix} -2 & -1 & 0 \\ 2 & -2 & 1 \\ 1 & 12 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 1 \\ 0.5 & -1 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} -1 & 1 & 0 \\ 2 & -2 & 1 \\ 1 & 1 & 0.5 \end{bmatrix},$$

$B = [\,0\;0\;1\,]^*$ and $C = [\,1\;0\;1\,]$.

Here matrices $\{A_{11}^i - A_{12}^i C_2^{-1} C_1\}$ are Hurwitz and normal. So by theorem 4.4, there exists a static output feedback controller. Using the algorithm given in section 4.5, we compute controller gain $L = 5$ and the Lyapunov matrix $W = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 3 \end{bmatrix}$.

It is also noticed that the submatrices $\{A_{11}^i\}$ are not same for all three systems. So these systems are different from matched systems.

4.6 Conclusion

In this chapter, three new classes of systems are ~~i~~dentified which are simultaneously quadratically stabilizable. An algorithm is also provided to compute such a controller. These classes of systems are different from the matched uncertain systems and considerably improve the results reported in the literature [27, 65]. Although these conditions are devised for linear systems using constant output feedback, they are equally applicable to linear systems using dynamic output feedback.

Chapter 5

Stabilization of Time Delay and Time Varying Systems

This chapter is divided into two parts. In the first part, sufficient conditions for memoryless state feedback stabilization of a set of time delay systems are presented. The results of chapter 2 and 3 are extended to stabilize the delay system. Lyapunov stability approach is employed to investigate this problem. Furthermore, these results also allow one to synthesize a controller using a simple algorithm.

In the second part, we discuss the usefulness of SQ stabilization problem for time varying systems. More specifically, we characterize certain classes of time varying systems which are stabilizable by state feedback controller. This chapter is organized as follows. In sections 5.1- 5.3, we discuss the stability of time delay systems. The results of time varying systems are discussed in section 5.4.

5.1 Preliminaries

In practice, time delay is commonly encountered in various engineering systems such as chemical processes, man-machine systems, long transmission lines in hydraulic and rolling mill systems etc. Hence the study of stability problem of time delay systems has drawn the attention of researchers over the past decade [39, 28, 82, 69, 17].

Consider a set of state delay system described by

$$\frac{dx(t)}{dt} = \dot{x}(t) = A_i x(t) + A_h x(t-h) + Bu \tag{5.1}$$

$$x(t) = \eta(t), t \in [-h, 0].$$

where $x \in R^n$ is the state vector, $u \in R^m$ is the control input and A_i, B, A_h are constant matrices. The parameter $h > 0$ is a scalar number representing the delay in the system and $\eta(t) \in C[-h, 0]$ is initial state vector-valued function which is continuous on the segment $-h \leq t \leq 0$. We represent $x_t = x(t+\theta)$ where $\theta \in [-h, 0]$. This set of systems will be denoted by $\{A_i, B, A_h\}$. It is also assumed that the solution of (5.1) exists. For more details regarding this see [39, 28, 82].

Sufficient conditions are derived in [10, 16, 28, 82] to stabilize a single Time Delay (TD) system (A_i, B, A_h) using memoryless state feedback controller of the form $u = -Kx(t)$. This type of controller requires knowledge of only instantaneous value of state $x(t)$. Using Riccati equation approach[1], the stabilization of uncertain system is solved in [11, 55, 61]. In this approach linear state feedback controller is sought to stabilize the delay system. The existence condition of these results are related to the solution of a parameter dependent Riccati equation. If the Riccati equation has some solution, then controller gain can be computed by their algorithm. It is also known that matched[2] uncertain system is asymptotically stabilizable by state feedback controller [61, 44].

In this part of the chapter, different class of delay systems are identified which will be stabilizable by a single memoryless state feedback controller. The state feedback controller, to be designed, is of the form

$$u = -Kx(t). \tag{5.2}$$

In order to design matrix $K \in R^{m \times m}$, we have combined the Lyapunov stability approach and some mathematical tools for the solution of the Riccati equation.

The following assumptions are made in this chapter about the family of systems described in (5.1).

[1]In this approach the uncertain TD system is described by $\dot{x}(t) = (A_0 + \Delta A(t))x(t) + (A_{h0} + \Delta A_h(t))x(t-h) + (B_0 + \Delta B(t))u$ where $\Delta A(t)$, $\Delta A_h(t)$, $\Delta B(t)$ represents perturbations. The system (A_0, A_{h0}, B_0) is the nominal system.

[2]The uncertain TD system (footnote 1 above) will satisfy matching conditions if [61, 44] $\Delta A(t) = B_0 O_1(t)$, $\Delta A_h = B_0 O_2(t)$, $\Delta B = B_0 O_3(t)$ where $\|O_3(t)\| < 1$.

Assumption 5.1 : The matrix B is assumed to be of the form $B = \begin{bmatrix} 0 \\ I_m \end{bmatrix}$.

Assumption 5.2 : The delay matrix $A_h = BD$ where matrix $D \in R^{m \times n}$.

Assumption 5.3 : It is assumed that the system pairs $\{A_i, B\}$ are stabilizable, i.e., the non delay systems are stabilizable.

In the assumption 5.1, the particular structure of B matrix is always possible using a similarity transformation and can be justified on the line of justification of assumption 2.1 (page 16).

5.2 Stability of Time Delay Systems

In this subsection, we shall discuss the stabilizability of a single TD system (A_i, B, A_h) by state feedback controller. Using the controller $u = -Kx(t)$, the closed loop system is written as

$$\dot{x}(t) = (A_i - BK)x(t) + A_h x(t-h) \tag{5.3}$$

To verify the stability of the closed loop system (5.3) (after plugging the controller), we consider a particular type of Lyapunov functional

$$V(x_t) = x^*(t)Px(t) + \int_{t-h}^{t} x^*(s)Qx(s)ds \tag{5.4}$$

where $P = P^* > 0$ and $Q = Q^* > 0$ are positive definite matrices. Notice that $V(x_t) > 0$ since $P > 0$ and $Q > 0$. Next we shall introduce the following definition regarding the restricted version of the stability of delay systems.

Definition 5.1 : The closed loop TD system (5.3) is quadratically stable if there exists a quadratic functional $V(x_t)$ of the form (5.4) and the derivative $\dot{V}(x_t)$ is negative definite along the trajectory of closed loop system.

Remarks 5.1 : For quadratically stable system, the solution $x(t)$ of closed loop system converges to $x(t) = 0$ as time $t \to \infty$ [28, 39].

With this brief introduction, we shall state a lemma for the stabilizability of a single TD system (A_i, B, A_h) by state feedback controller. In this context, a delay system

will be stabilizable by state feedback controller if the corresponding closed loop system is quadratically stable.

Lemma 5.1 *A time delay system (A_i, B, A_h) is memoryless state feedback stabilizable via the Lyapunov functional (5.4) if there exists a memoryless controller $u = -Kx(t)$ and a positive definite matrix P such that the following inequality holds :*

$$(A_i - BK)^*P + P(A_i - BK) + PA_hQ^{-1}A_h^*P + Q < 0.$$

Proof : Choose the Lyapunov functional as given in (5.4)

$$V(x_t) = x^*(t)Px(t) + \int_{t-h}^{t} x^*(s)Qx(s)ds.$$

Defining $\bar{A}_i = A_i - BK$, the derivative of $V(x_t)$ along the solution of closed loop system is

$$\begin{aligned}
\dot{V}(x_t) &= x^*(t)(\bar{A}_i^*P + P\bar{A}_i + Q)x(t) \\
&\quad + x^*(t-h)A_h^*Px(t) + x^*(t)PA_hx(t-h) - x^*(t-h)Qx(t-h) \\
&= \begin{bmatrix} x^*(t) & x^*(t-h) \end{bmatrix} \begin{bmatrix} \bar{A}_i^*P + P\bar{A}_i + Q & PA_h \\ A_h^*P & -Q \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix} \\
&= \begin{bmatrix} x^*(t) & x^*(t-h) \end{bmatrix} [THT^*] \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}.
\end{aligned}$$

Here matrix $T = \begin{bmatrix} I_n & -PA_h \\ 0 & Q \end{bmatrix}$, $H = \begin{bmatrix} \bar{A}_i^*P + P\bar{A}_i + PA_hQ^{-1}A_h^*P + Q & 0 \\ 0 & -Q^{-1} \end{bmatrix}$.

Since matrix T is nonsingular, $\dot{V}(x_t)$ will be negative definite if $H < 0$. So delay system will be stable if

$$(A_i - BK)^*P + P(A_i - BK) + PA_hQ^{-1}A_h^*P + Q < 0 \tag{5.5}$$

□

Lemma 5.1 provides a delay independent stability criteria for a single system (A_i, B, A_h) since the above inequality does not include the delay h.

In this chapter, we use $Q = I_n$ in the general expression of $V(x_t)$ in (5.4). So using $Q = I_n$ in (5.5), the following definition is introduced here.

Definition 5.2 : A set of TD systems $\{A_i, B, A_h\}$ is SQ stabilizable if there exists a matrix K and a matrix $P > 0$ such that the following hold

$$(A_i - BK)^*P + P(A_i - BK) + PA_hA_h^*P + I_n < 0 \; \forall \, i.$$

Next we shall state the simultaneous stabilization of a set of TD system $\{A_i, B, A_h\}$ in the form of following lemma

Lemma 5.2 *A set of systems $\{A_i, B, A_h\}$ defined in (5 1) is simultaneously quadratically stabilizable by a state feedback controller if there exists a scalar γ and a positive definite matrix W such that the following inequalities holds*

$$A_i W + W A_i^* - \gamma B B^* + A_h A_h^* + W W < 0 \,\forall\, i. \tag{5.6}$$

Proof : Suppose there exists a matrix W and γ such that (5.6) holds. Now pre- and post-multiplying (5.6) by W^{-1}, we get

$$(A_i - BK)^* P + P(A_i - BK) + P A_h A_h^* P + I_n < 0 \,\forall\, i \tag{5.7}$$

where $P = W^{-1}$ and

$$K = (\gamma/2) B^* W^{-1}. \tag{5.8}$$

Next choose a Lyapunov functional as given in (5.4) with $Q = I_n$ which yields

$$V(x_t) = x^*(t) P x(t) + \int_{t-h}^{t} x^*(s) x(s) ds.$$

The derivative of $V(x_t)$ will be negative definite for each system (A_i, B, A_h) in the light of inequalities (5.7) and lemma 5.1. So each system is stabilizable by a control law $u = -Kx$.
□

It is observed that if (5.6) holds, then simultaneous stabilizing controller is given by $u = -(\gamma/2) B^* W^{-1} x$. Next the following partition has been made in this section.

$$A_i = \begin{bmatrix} A_{11}^i & A_{12}^i \\ A_{21}^i & A_{22}^i \end{bmatrix}, B = \begin{bmatrix} 0 \\ I_m \end{bmatrix}, A_h = \begin{bmatrix} 0 \\ D \end{bmatrix} \tag{5.9}$$

where $A_{11}^i \in R^{n-m \times n-m}, A_{12}^i \in R^{n-m \times m}, D \in R^{m \times n}$.

5.3 Main Results

Using above partition (5.9), the following result is derived.

Theorem 5.1 *Suppose the system (5.1) satisfies the assumptions 5.1-5.3 and matrices $\{A_i, B, A_h\}$ are partitioned as given in (5.9). These system are simultaneously quadratically stabilizable if there exists a matrix $W_1 = W_1^* > 0$ and a matrix F such that the following inequalities hold*

$$(A_{11}^i + A_{12}^i F)W_1 + W_1(A_{11}^i + A_{12}^i F)^* + W_1(I_{n-m} + F^*F)W_1 < 0 \; \forall \; i \tag{5.10}$$

Proof : The proof is similar to the proof of theorem 2.1 of chapter 2 and can be found in Appendix B. □

Remarks 5.2 : In theorem 5.1, we assume the matrices B and A_h are constant for all the systems. However, we can derive similar results for a set of TD systems $\{A_i, B_i, A_{hi}\}$ where

$$B_i = BG_i, \quad (G_i + G_i^*) > 0 \text{ and} A_{hi} = BD_i$$

The details of these issues are not discussed here.

The condition derived in the above theorem is not appealing enough because it involves two additional matrices W_1 and F. However, it is expected that the condition should be derived in terms of given state space parameters A_i, B, and A_h. For this purpose, the following class of uncertain TD systems are considered for the solution of SQ stabilization problem. These systems are (1) **matched systems**, (2) **partially commutative systems** and (3) **partially normal systems**. Each of these family of systems satisfies the assumptions (5.1-5.3). These classes of systems are defined in the context of finite number of TD systems as follows and similar in definition given in chapter 3.

Definition 5.2 : A set of TD system $\{A_i, B, A_h\}$ is said to be matched system if the submatrices A_{11}^i and A_{12}^i (defined in (5.9)) are same for each system.

Definition 5.3 : A set of TD system $\{A_i, B, A_h\}$ is said to be partially commutative if the submatrices $\{A_{11}^i\}$ are Hurwitz and pairwise commutative.

Definition 5.4 : A set of TD system $\{A_i, B, A_h\}$ is said to be partially normal if the submatrices $\{A_{11}^i\}$ are Hurwitz and normal.

Defining the above classes of systems, we shall state three theorems for the solution SQ stabilization problem of these class of systems.

Theorem 5.2 *A set of matched TD systems $\{A_i, B, A_h\}$ is simultaneously quadratically stabilizable by a single memoryless feedback controller.*

Proof : Based on the definition 5.2, we define and partitioned the matched systems $\{A_i, B, A_h\}$ as follows :

$$A_i = \begin{bmatrix} A_{11} & A_{12} \\ A_{21}^i & A_{22}^i \end{bmatrix}, B = \begin{bmatrix} 0 \\ I_m \end{bmatrix}, A_h = \begin{bmatrix} 0 \\ D \end{bmatrix}.$$

It is noticed that the non delay systems $\{A_i, B\}$ are stabilizable, so (A_{11}, A_{12}) is also stabilizable by lemma 2.3. Thus there exists a matrix F such that $(A_{11}+A_{12}F)$ is Hurwitz. Using such a matrix F, solve for $P_1 = P_1^* > 0$ from the following Lyapunov equation

$$(A_{11} + A_{12}F)^* P_1 + P_1(A_{11} + A_{12}F) + (I_{n-m} + F^*F) + Q = 0 \; \forall \; i$$

where Q is any positive definite matrix. By pre- and post-multiplying above equation by matrix $W_1 = P_1^{-1}$, the condition given in (5.10) is satisfied. Hence the proof. □.

Theorem 5.3 *A set of partially commutative TD systems $\{A_i, B, A_h\}$ is simultaneously quadratically stabilizable.*

Proof : To prove this theorem, take $F = 0$. So (by substituting $F = 0$ in (5.10)) the problem is to find a matrix $W_1 = W_1^* > 0$ such that the following inequalities hold

$$A_{11}^i W_1 + W_1(A_{11}^i)^* + W_1 I_{n-m} W_1 < 0 \; \forall \; i. \tag{5.11}$$

By hypothesis, the matrices $\{A_{11}^i\}$ are Hurwitz and pairwise commutative. Hence by lemma 3.1 or 3.2 of chapter 3 (page 39), there exists a matrix $P_1 = P_1^* > 0$ such that

$$A_{11}^i P_1 + P_1(A_{11}^i)^* < 0 \; \forall \; i.$$

After computing such a matrix P_1, define matrix $W_1 = \beta P_1$ where $\beta > 0$ is a positive

scalar. Next, we shall show that there exists a β such that (5.11) holds for a matrix $W_1 = \beta P_1$. For this purpose, define a set of positive definite matrices $\{Q_i\}, i = 1, 2, \ldots, l$ where $Q_i = -[\, A^i_{11} P_1 + P_1 (A^i_{11})^* \,]$.

By substituting this $W_1 = \beta P_1$ in (5.11), we obtain

$$\beta [\, -Q_i + \beta P_1 P_1 \,] < 0 \; \forall \; i \tag{5.12}$$

It is evident from the above expression (5.12) that each matrix $-Q_i$ is a negative definite matrix and $P_1 P_1$ is a positive definite matrix. So by taking sufficiently small positive value of β, (5.12) will be satisfied. The value of β is computed as follows.
For each index i, define the scalar β_i such that the following inequality holds

$$\beta_i [\, -Q_i + \beta_i P_1 P_1 \,] < 0.$$

One value of such β_i is computed by the formula

$$\beta_i < \lambda_m[Q_i] / \lambda_M[P_1 P_1]$$

where $\lambda_m[.]$ is the minimum eigenvalue of $[.]$. The required value of β to satisfy (5.12) is selected as

$$\beta = \min\{\beta_i\}, i = 1, 2, \ldots, l.$$

This completes the proof of theorem. □

Theorem 5.4 *A set of partially normal TD systems $\{A_i, B, A_h\}$ is simultaneously quadratically stabilizable.*

Proof : Using the lemma 3.3 (page 40), the proof is similar to that of theorem 5.3. □

Remarks 5.3 : So far we have concentrated on designing state feedback controller for TD systems. The static output feedback controller design can be carried for TD systems in the same way as the results of chapter 3. However we have not discussed these issues in detail.

5.4 Controller Design for Time Varying System

In this section constant feedback controller design for time varying system is presented. In the previous chapters, we have identified different classes of systems for which SQ stabilization problem is solvable based on Lyapunov stability theory. Here we derive the stability results for different classes of time varying systems using the SQ stabilization results of previous chapters. This is discussed below

Consider a linear time varying system $(A(t), B(t))$ described by

$$\dot{x}(t) = A(t)x(t) + B(t)u \tag{5.13}$$

The problem considered in this section is to design a static state feedback controller $u = -Kx(t)$ such that the equilibrium state of the closed loop time varying system

$$\dot{x}(t) = (A(t) - B(t)K)x(t) \tag{5.14}$$

is exponentially stable.

Given two sets of constant matrices $\{A_i\}, i = 1, 2, \dots, l$ and $\{B_i\}, i = 1, 2, \dots, l$, define a parametrized family of systems described by state space equation

$$\dot{x}(t) = \sum_{i=1}^{l} \alpha_i(t)A_i x(t) + \sum_{i=1}^{l} \alpha_i(t)B_i u = A_\alpha x + B_\alpha u \tag{5.15}$$

where $\alpha_i(t) \geq 0$ is a piecewise continuous scalar function and $\sum_{i=1}^{l} \alpha_i(t) > 0$ for all $t \geq 0$. In this case matrix B_i satisfies assumption 2.3, i.e., $B_i = BG_i$ where $B = \begin{bmatrix} 0 \\ I_m \end{bmatrix}$ and $(G_i + G_i^*) > 0$. This parameterized family is represented by the following set S as

$$S = \{(A_\alpha, B_\alpha) : A_\alpha = \sum_{i=1}^{l} \alpha_i(t)A_i, B_\alpha = \sum_{i=1}^{l} \alpha_i(t)B_i, \alpha_i(t) \geq 0 \;\forall\, i \text{ and} \sum_{i=1}^{l} \alpha_i(t) > 0\}.$$

For different instant of time, $\alpha_i(t)$ takes different values. Suppose, a time varying system $(A(t), B(t)) \in S$ for all time t, then matrices $A(t)$ and $B(t)$ are written as

$$A(t) = \sum_{i=1}^{l} \alpha_i(t)A_i, \; B(t) = \sum_{i=1}^{l} \alpha_i(t)B_i \;\forall\, t$$

The question is whether there exists a single control law $u = -Kx$ such that the time varying systems (5.13) along with the control law is exponentially stable. To give an answer of this question, we next introduce the following definition.

Definition 5.2 : The linear time varying system (5.13) along with a control law $u = -Kx(t)$ is exponentially stable if there exists a **quadratic** Lyapunov function $V(x) = x^*Px$, with $P > 0$, such that the $\dot{V}(x) < 0$ along the solution of closed loop system.

Remarks 5.2 : For exponentially stable time varying system, the solution of closed loop system (5.14) will converge to **0** as $t \to \infty$ [73].

Next we shall state the stabilization results for time varying systems.

Theorem 5.5 *The time varying system (5.13) along with a state feedback control law* $u = -Kx$ *is exponentially stable if the system realization* $(A(t), B(t)) \in S \;\; \forall\, t \geq 0$ *and the matrices* $\{A_i\}$ *and* $\{B_i\}$ *satisfy any one of the following conditions.*
(a) $\{A_i, B_i\}$ *are matched systems*

or

(b) $\{A_i, B_i\}$ *are simultaneously transformable into Hessenberg form*

or

(c) $\{A_i, B_i\}$ *are partially commutative or partially normal family.*

Proof : The theorem can be proved by employing the results of chapter 2 and chapter 3. Here we provide the proof of the part (a) and the proofs of other parts will be similar.
proof of (a) : Since the systems $\{A_i, B_i\}$ are matched systems then there exists a controller gain K and matrix $W > 0$ such that the following inequalities hold by theorems 2.3 and 2.7

$$(A_i - B_iK)^*W^{-1} + W^{-1}(A_i - B_iK)^* < 0 \;\forall\, i = 1, 2, \ldots, l \tag{5.16}$$

Now consider the quadratic Lyapunov function $V(x) = x^*W^{-1}x$. The derivative of this function along the solution of closed loop system is

$$\begin{aligned} \dot{V}(x) &= x^*(t)[\,(A(t) - B(t)K)^*W^{-1} + W^{-1}(A(t) - B(t)K)\,]x(t) \\ &= x^*(t)(\sum_{i=1}^{l} \alpha_i(t)[\,(A_i - B_iK)^*W^{-1} + W^{-1}(A_i - B_iK)^*\,])x(t) \end{aligned}$$

Using (5.16) and noting that positive combination of negative definite matrices are negative definite, $\dot{V}(x) < 0 \;\forall\, t \;\geq\; 0$. This proves the exponential stability of the time varying system (5.13) with a state feedback control law. □

Definition 5.2 : The linear time varying system (5.13) along with a control law $u = Kx(t)$ is exponentially stable if there exists a **quadratic** Lyapunov function $V(x) = x^* P x$, with $P > 0$, such that the $\dot{V}(x) < 0$ along the solution of closed loop system.

Remarks 5.2 : For exponentially stable time varying system, the solution of closed loop system (5.14) will converge to 0 as $t \to \infty$ [73].

Next we shall state the stabilization results for time varying systems.

Theorem 5.5 *The time varying system (5.13) along with a state feedback control law $u = -Kx$ is exponentially stable if the system realization $(A(t), B(t)) \in S \;\; \forall\, t \geq 0$ and the matrices $\{A_i\}$ and $\{B_i\}$ satisfy any one of the following conditions.*

(a) $\{A_i, B_i\}$ are matched systems

or

(b) $\{A_i, B_i\}$ are simultaneously transformable into Hessenberg form

or

(c) $\{A_i, B_i\}$ are partially commutative or partially normal family.

Proof : The theorem can be proved by employing the results of chapter 2 and chapter 3. Here we provide the proof of the part (a) and the proofs of other parts will be similar.

proof of (a) : Since the systems $\{A_i, B_i\}$ are matched systems then there exists a controller gain K and matrix $W > 0$ such that the following inequalities hold by theorems 2.3 and 2.7

$$(A_i - B_i K)^* W^{-1} + W^{-1}(A_i - B_i K)^* < 0 \;\forall\, i = 1, 2, \ldots, l \qquad (5.16)$$

Now consider the quadratic Lyapunov function $V(x) = x^* W^{-1} x$. The derivative of this function along the solution of closed loop system is

$$\begin{aligned} \dot{V}(x) &= x^*(t)[\,(A(t) - B(t)K)^* W^{-1} + W^{-1}(A(t) - B(t)K)\,]x(t) \\ &= x^*(t)(\sum_{i=1}^{l} \alpha_i(t)[\,(A_i - B_i K)^* W^{-1} + W^{-1}(A_i - B_i K)^*\,])x(t) \end{aligned}$$

Using (5.16) and noting that positive combination of negative definite matrices are negative definite, $\dot{V}(x) < 0 \;\forall\, t \;\geq\; 0$. This proves the exponential stability of the time varying system (5.13) with a state feedback control law. □

5.5 Conclusion

In the first part of this chapter, the controller design problem for a collection of time delay system is discussed. Different classes of delay systems are shown to be stablizable by a single static state feedback controller. These results are valid for matched uncertain system as well as certain class of mismatched systems. These mismatch classes of systems can be thought of as a major contribution in this direction compared to the existing literature where the existence conditions of only matched time varying uncertain systems have been ensured. These results can be extended using memoryless output feedback controller using the results of chapter 3.

Deviating from time delay systems, the exponential stability results of linear time varying systems are considered in section 5.4. The SQ stabilization results of LTI system is successfully utilized to design a single constant feedback controller for special classes of time varying systems.

Chapter 6

Applications to Case Studies

In this chapter, theory developed in this thesis has been applied to some practical systems. For this purpose, we consider two examples, namely Aircraft control and Robot trajectory control. In the first part of this chapter, we discuss the feedback controller design of an aircraft model which is uncertain. In the second part of this chapter, the trajectory control of robot manipulator has been discussed.

6.1 Robust Stabilization of an Aircraft

Consider the 4th order model [40] of the longitudinal motion of an aircraft described by

$$\dot{x} = A(r)x + Hu \tag{6.1}$$

where $A(r) = \begin{bmatrix} 0.0 & 1.0 & 0.0 & 0.0 \\ 1.4010 \times 10^{-4} & r & -1.9513 & 0.0133 \\ -2.5050 \times 10^{-4} & 1.0 & -1.3239 & -0.0 \\ -0.561 & 0.0 & 0.3580 & -0.0279 \end{bmatrix}$ and

$$H = \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ -5.3307 & 6.447 \times 10^{-3} & -0.2667 \\ -0.1600 & -1.155 \times 10^{-2} & -0.2511 \\ 0.0 & 0.1060 & 0.0862 \end{bmatrix}.$$

The state vector are defined as $x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^*$ and control input as $u = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^*$.

The state variables and control inputs are defined as follows :

x_1 : pitch attitude

x_2 : pitch rate

x_3 : angle of attack

x_4 : air speed

u_1 : elevator angle

u_2 : throttle control

u_3 : flap deflection.

In the above equation (6.1), $A(r)$ matrix is uncertain and H matrix is fixed. The uncertain element r varies from -0.558 to -3.558.

Define new set of control inputs $v = Gu$ where the nonsingular matrix G is given by

$$G = \begin{bmatrix} -5.3307 & 6.447 \times 10^{-3} & -0.2667 \\ -0.1600 & -1.155 \times 10^{-2} & -0.2511 \\ 0.0 & 0.1060 & 0.0862 \end{bmatrix}.$$

Using this new set of control inputs, we define a set of two extreme plants $\{A_i, B\}$ described as

$$\dot{x} = A_i x + Bv \tag{6.2}$$

where $B = \begin{bmatrix} 0 \\ I_m \end{bmatrix}$. In the above equation matrix $A_1 = A(r)$ when $r = -0.558$ and $A_2 = A(r)$ when $r_2 = -3.558$. These two systems may belong to matched system or a general family of systems defined in chapter 3. Using the algorithm of section 2.9 (page 33) we compute the Lyapunov matrix as $W = \begin{bmatrix} 1 & -1.1 & 0.5 & 0.35 \\ -1.1 & 2.46 & -0.35 & -0.235 \\ 0.5 & -0.35 & 1.41 & 0.295 \\ 0.35 & -0.235 & 0.295 & 1.212 \end{bmatrix}$.

Using the model $\{A_i, B\}$, the stabilizing controller gain is computed as

$$\bar{K} = \begin{bmatrix} 1.2729 & 1.0417 & -0.1667 & -0.1250 \\ -0.7067 & -0.1667 & 1.1167 & -0.1000 \\ -0.4988 & -0.1250 & -0.1000 & 1.1750 \end{bmatrix}.$$

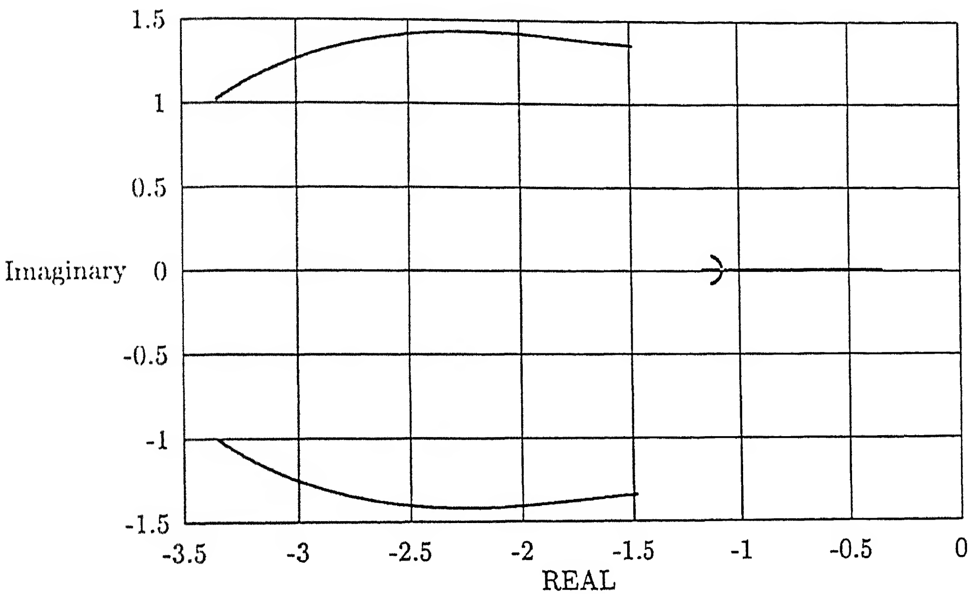

Figure 1: Distribution of eigenvalues of $A(r) - HK$ with variation in r

The actual control law for the stabilization of $\{A_i, H\}$ is given by $u = G^{-1}\bar{K}x$ where

$$K = \begin{bmatrix} -0.4193 & -0.2431 & 0.2727 & 0.0443 \\ -7.4912 & -1.9167 & 2.9237 & 11.203 \\ 3.4260 & 0.9068 & -4.7554 & -0.1453 \end{bmatrix}.$$

It must be noticed from (6.1) that single uncertain element r varies between -0.558 and -3.558. The actual uncertain system $(A(r), H)$ can be written as the convex combination of two systems (A_1, H) and (A_2, H). More specifically,

$$A(r) = \sum_{i=1}^{2} \alpha_i A_i, H = \alpha_i H, \alpha_i \geq 0, \sum_{i=1}^{2} \alpha_i = 1.$$

So the uncertain system $(A(r), H)$ is stabilizable by a single control law $u = -Kx$. To verify this, we plot in figure 1 the distribution of the eigenvalues of matrix $(A(r) - HK)$ with the variation of r in the range -0.558 to -3.558. This confirms that all eigenvalues of matrix $(A(r) - HK)$, with $-3.558 \leq r \leq -0.558$, have negative real part.

6.2 Controller Design For Robot Manipulator

It has been shown in chapter 2 that a set of systems in block companion form is simultaneously quadratically stabilizable by a single static state feedback controller. Based on this result, a new approach to robot tracking controller design is presented. The proposed control scheme consists of a feedforward controller based on the inverse dynamics of the robot and a feedback controller. The non-linear model of the robot is viewed as piecewise LTI systems obtained by linearizing the model at selected number of points on a specified trajectory in the joint space. The collection of all the LTI systems constitutes a set in which each member is observed to be in block companion form. For such class of systems, an algorithm for the design of a single stabilizing feedback controller is presented. A numerical example of a two link manipulator is considered to validate the proposed theory.

6.3 Preliminaries

Present-day control literature reveals umpteen number of research works on the tracking control of non-linear systems. Of the many approaches available, the application of linear multi-variable theory has received wide acceptance because of its simplicity and ease of understanding. In this approach, the non-linear system is viewed as a set of LTI systems over some specified trajectory. In such a situation, it is always desirable to have a single stabilizing controller for the above mentioned set of LTI systems and hence for the overall non-linear system. This motivates the application of the simultaneous stabilization problem which amounts to the design of a single feedback controller for stabilizing a finite number of plants.

The problem of control of robot manipulators for trajectory tracking has long attracted the attention of researchers and engineers (see, for example [37, 63, 19] and the references therein). This problem is concerned with the design of an appropriate control scheme to track time varying joint space or task space trajectories. In this chapter we shall focus only on the former. There are, at present, numerous approaches to the design of robot controllers of which the *Computed Torque Technique* (CTT) is most common. The CTT scheme is based on the global linearization of the robot dynamics by means of a non-linear feedback controller. However, the order of complexity of the controller is same as that of the robot model [52] which makes it computationally expensive. Seraji [60] proposed

a simple linear control scheme using a feedforward and state variable feedback control. The robot dynamics is approximated by a set of linearized models over the trajectory. However, the control scheme requires the updating of feedback gains which can be realized by off-line or on-line computation with gain scheduling. This process of updating results in the overall system to be time varying and may lead to deterioration of performance due to switching of the gains depending on its rate. Moreover, for off-line gain scheduling, there is a trade-off between the performance of the controller and the memory requirement for storing the the pre-computed gains. Similar control scheme has also been proposed by Swarup and Gopal [66] using a performance optimization with eigen structure assignment. They also considered the uncertain dynamics of a manipulator to solve tracking problem.

In chapter 2, we have presented an algorithm for the construction of a *single* static state feedback controller for a set of LTI systems This feedback controller is designed based on the construction of a single Lyapunov function for the set of LTI systems and its applicability to time varying systems is discussed in chapter 5 (section 5.4). The theory developed so far is then used to design a feedforward and a single static state-variable feedback controller for a serial chain robot manipulator. The feedforward controller ensures the tracking of the manipulator joint trajectory while, the feedback controller ensures the exponential stability of the overall closed loop dynamics. The proposed approach has a distinct advantage over the previous approaches [60] in that we do not require any feedback gain scheduling.

6.4 Application to Robot Control

In this section, the theory developed thus far are applied to the problem of robot control. The problem considered here is to develop a control scheme which ensures that the joint angle vector $\theta(t)$ tracks any desired (reference) trajectory $\theta_r(t)$ where $\theta_r(t)$ is a vector of smooth time functions. A simple linear control design consisting of a feedforward and a static state-variable feedback control is proposed.

6.4.1 Robot Dynamics : Linearized Model

Using the Lagrange-Euler formulation, the equations of motion of a serial-chain manipulator can be derived in the following form [19, 37, 63]

$$M(\theta)\ddot{\theta} + N(\theta, \dot{\theta}) + G(\theta) = T \tag{6.3}$$

where $M(\theta)$ is $n \times n$ symmetric positive definite inertia matrix, $N(\theta, \dot{\theta})$ is $n \times 1$ Coriolis and centripetal forces, $G(\theta)$ is $n \times 1$ gravity loading vector and T is the $n \times 1$ vector of joint torques. $\theta, \dot{\theta}, \ddot{\theta}$ are $n \times 1$ vectors of joint angles, velocities and accelerations, respectively. The elements of $M(\theta), N(\theta, \dot{\theta})$ and $G(\theta)$ are highly non-linear functions of the joint configurations, velocities and the payload.

After the desired trajectory $\theta_r(t)$ has been determined in the planning stage, the appropriate torque is required to result in the desired motion. Under the ideal conditions (the model is accurate and the absence of any disturbances), the desired torque is given by [37]

$$T^d = M(\theta_r)\ddot{\theta}_r + N(\theta_r, \dot{\theta}_r) + G(\theta_r).$$

If this torque is applied to the manipulator modeled by (6.3), the resulting motion tracks the desired trajectory under ideal circumstances. Because of disturbances and inaccuracies in the system model, the actual motion of the manipulator usually deviates from the desired trajectory. These undesirable deviations can be corrected by means of feedback controller. So using PD type feedback controller, the total torque which can be used for the tracking purpose is

$$T = T^d + K_p(\theta_r - \theta) + K_v(\dot{\theta}_r - \dot{\theta}).$$

where K_p and K_v are the appropriate feedback gain.

Since the actual motion of manipulator deviates from the nominal trajectory, it is necessary to study deviation versus time behaviour from the nominal trajectory and the compensation for perturbations. These goals can be achieved by using linearized model of the manipulator dynamics that describes the small perturbations about the nominal trajectory. The determination of such linear model is then discussed which will be used for the purpose of controller design.

In the following developments, a specified trajectory in the joint space is considered as a collection of points lying on it. Consider the i^{th} nominal point (node point) denoted by $S^i = (\theta_{ni}, \dot{\theta}_{ni}, \ddot{\theta}_{ni}, T_{ni})$. The nominal joint angle vector θ_{ni} and its higher derivatives are

taken on the prespecified trajectory and T_{ni} is the corresponding nominal torque. At the time of linearization, it has been assumed that the inertia matrix does not change appreciably for a small deviation x about the nominal point, i.e., $M(\theta_{ni} + x(t)) \approx M(\theta_{ni})$ [60]. The linearized dynamic model, with respect to the i^{th} nominal point, may be represented in the following form

$$N_{2i}\ddot{x} + N_{1i}\dot{x} + N_{0i}x = u \tag{6.4}$$

where N_{2i}, N_{1i}, N_{0i} are $n \times n$ constant matrices obtained as

$$N_{2i} = [M(\theta)]_{S^i}, N_{1i} = \left[\frac{\partial N}{\partial \dot{\theta}}\right]_{S^i}, N_{0i} = \left[\frac{\partial (N+G)}{\partial \theta}\right]_{S^i}.$$

The vectors x and u are deviations from the nominal values of joint positions and torques, respectively. Defining the state variables as $(x, \dot{x})$, the above linearized equation (6.4) is represented in the state space form as

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -N_{2i}^{-1}N_{0i} & -N_{2i}^{-1}N_{1i} \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ N_{2i}^{-1} \end{bmatrix} u. \tag{6.5}$$

It may be noted that the robot model over the trajectory is a slowly time varying system [19. 60] and is approximated by piecewise (in time) LTI in nature. The matrices N_{2i}, N_{1i}, N_{0i} are evaluated at different node points $i = 1, 2, \ldots$ and are used to obtain a set of LTI systems in the form of (6.5).

6.4.2 Controller Design

In this section, a control scheme is presented based on the linearized model obtained in (6.4). As may be noted from (6.5), the linearized dynamics is always controllable and the matrix $N_{2i}^{-1} > 0$. N_{2i} being the inertia matrix of the robot evaluated at a nominal point.

At a nominal point on the trajectory, all the non-linearities in the robot dynamics are exactly canceled by the nominal joint torque T_{ni} since it is calculated based on the non-linear inverse dynamics (6.3) at the nominal point. About this point, the linearized dynamics is represented by (6.4). The proposed control scheme consists of feedforward and feedback controllers based on this linearized model. The implementation of the scheme for the non-linear dynamics of the robot is shown in Fig. 2 where s is the differential operator.

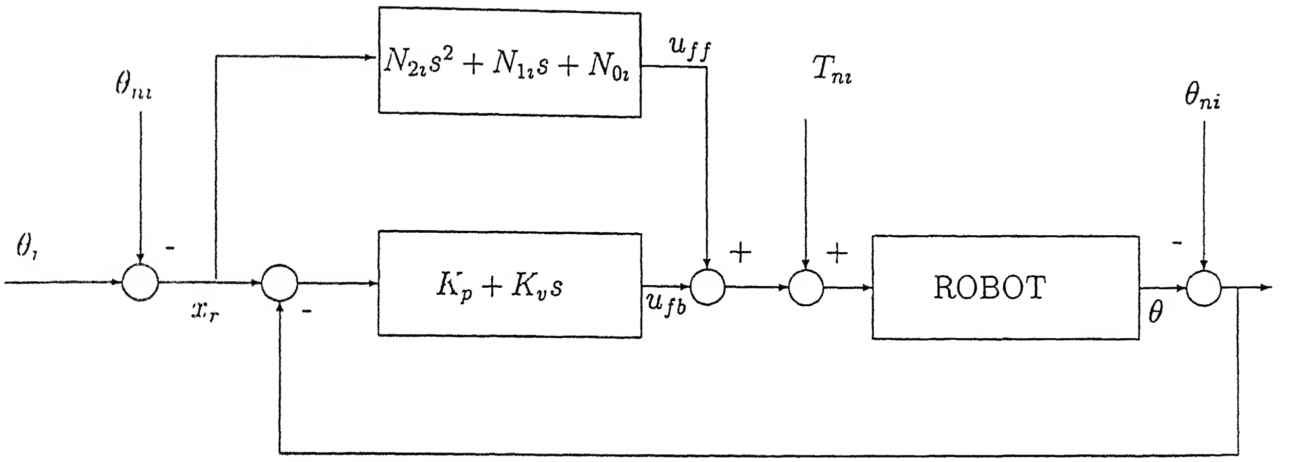

Figure 2: Structure of Robot Control Systems

The total control law is given by

$$U(t) = u_{ff}(t) + u_{fb}(t) + T_{ni} = u(t) + T_{ni} \tag{6.6}$$

where, $u_{ff}(t)$ and $u_{fb}(t)$ are the feedforward and feedback controllers respectively and $u(t) = u_{ff}(t) + u_{fb}(t)$.

The feedforward controller, computed based on the inverse dynamics of the linearized robot model (6.4), provides tracking ability and is given by [60]

$$u_{ff}(t) = N_{2i}\ddot{x}_r + N_{1i}\dot{x}_r + N_{0i}x_r. \tag{6.7}$$

where x_r denotes the deviation of the nominal point on the reference trajectory, i.e., the reference vector $\theta_r(t) = \theta_{ni} + x_r$.

The feedback controller provides the asymptotic stability of the overall system and is discussed below. Substitution of the expressions for $u(t)$ in (6.4) yields the closed loop dynamics as

$$\begin{aligned} N_{2i}\ddot{x} + N_{1i}\dot{x} + N_{0i}x &= N_{2i}\ddot{x}_r + N_{1i}\dot{x}_r + N_{0i}x_r + u_{fb}(t) \\ \Rightarrow \quad N_{2i}\ddot{e} + N_{1i}\dot{e} + N_{0i}e &= -u_{fb}(t) \end{aligned} \tag{6.8}$$

where, $e(t) = x_r(t) - x(t)$ is the tracking error.

Introducing the state vector $\eta = [\ e(t)^* \quad \dot{e}(t)^*\ ]^*$, (6.8) can be written as

$$\dot{\eta}(t) = A_i\eta(t) + B_i u_{fb}(t) \tag{6.9}$$

where, $A_i = \begin{bmatrix} 0 & I_n \\ -N_{2i}^{-1}N_{0i} & -N_{2i}^{-1}N_{1i} \end{bmatrix}$ and $B_i = \begin{bmatrix} 0 \\ -N_{2i}^{-1} \end{bmatrix}$.
As the robot manipulator moves over the reference trajectory, the A_i and B_i matrices change continuously resulting in a linear, slowly time varying system [19]. Hence, on a discretized trajectory as in the present case, the linearized dynamics may be approximated by piecewise constant (in time) A_i and B_i matrices [60]. Asymptotic tracking is achieved here in the sense that $\eta(t) \to 0$ as $t \to \infty$.

At this point a discussion on the previous research is in order. In the existing approaches, the state feedback controller of the form $u_{fb}(t) = K_i\eta(t)$ has been considered. The matrix K_i is computed at every nominal point S^i on the reference trajectory such that, all eigenvalues of the closed loop matrix $(A_i + B_iK_i)$ are in the open left half plane and remain unchanged. The gain matrix K_i is *updated* successively as the manipulator moves. This calls for a scheduling of the feedback gains which may be computation intensive and may deteriorate tracking performance depending on the switching frequency [60, 70]. This motivates us to propose a new design methodology in which a *single* feedback controller is designed to ensure the exponential stability of the tracking error dynamics of the robot over a specified trajectory. Using a state feedback controller of the form $u_{fb}(t) = K\eta(t) = K_pe(t) + K_v\dot{e}(t)$, the overall closed loop error dynamics over the trajectory is of the form

$$\dot{\eta}(t) = A_c(t)\eta(t) \tag{6.10}$$

where, $A_c(t) = (A_i + B_iK), t_i \leq t < t_{i+1}$. The time instants t_i and t_{i+1} correspond to two consecutive nominal points, S_i and S_{i+1}, respectively. The key steps in the design procedure are :

- Finding a feedback controller of the form $u_{fb}(t) = K\eta(t)$.
- Finding a quadratic Lyapunov function of the form $v(\eta) = \eta^*P\eta$ such that $\dot{v}(\eta) < 0$ along the solutions of (6.10).

It is observed from (6.9) that the error dynamics obtained from the linearized model (6.5) is in block companion form. The collection of all such systems, defined on the trajectory,

constitutes a set of LTI systems $\{A_i, B_i\}$. It is also noted from the expression of B_i matrix of (6.9) that the matrix $N_{2i}^{-1} > 0$, N_{2i} being the inertia matrix of the robot evaluated at a nominal point. This allows us to use the result of theorem 2.5 and the subsequent algorithm for designing a single state feedback controller gain K. Further, a common Lyapunov matrix $P > 0$ exists such that

$$(A_i + B_i K)^* P + P(A_i + B_i K) < 0, \forall\, i. \tag{6.11}$$

Next, to show the exponential stability of (6.10), consider the Lyapunov function $v(\eta) = \eta^* P \eta$. Taking the time derivative of $v(\eta)$ and using (6.11) yields

$$\dot{v}(\eta) = \eta^* [A_c(t)^* P + P A_c(t)] \eta < 0 \;\forall\, t.$$

Thus, the asymptotic tracking is achieved in the sense that the tracking error $\eta(t) \to 0$ as $t \to \infty$.

Remarks 6.1 : In the above discussions, the exponential stability of linearized error dynamics is proved. However, the error dynamics for actual nonlinear systems is also stable locally with respect to the given trajectory.

Remarks 6.2 : In this example, we have not considered the uncertainty of parameters or load. The complete investigation should be done in this direction. However, our motive behind taking this example is to show the application of SQ stabilization results to a class of linear time varying systems.

6.4.3 Numerical Example

In order to validate the theory presented, a numerical example of a two-link planar manipulator, as shown in Fig. 2, is considered. The inertial and geometric parameters of the manipulator are taken from [60]. For the purpose of simulation, the manipulator is assumed to move from an initial joint angle configuration of $[\pi/2,\ 0]$ to the final configuration of $[0,\ \pi/2]$. The initial and final joint angle velocities are assumed to be zero. This is achieved by planning the reference trajectory in the following functional form

$$\begin{aligned}
\theta_{r1}(t) &= \pi/2[3exp(-t/0.3) - 4exp(-t/0.4)], \\
\theta_{r2}(t) &= \pi/2[1 + 3exp(-t/0.3) - 4exp(-t/0.4)]
\end{aligned}$$

where t varies from 0-5 sec. The exponential form of the trajectories are chosen to ensure smoothness. The reference trajectory is discretized by choosing 50 equidistant node points at which the dynamics is linearized to obtain a set of LTI systems. Following the algorithm presented in section (2.9), a single state feedback controller is designed as

$$K = \begin{bmatrix} -87.43 & 10.43 & -124.0 & -0.29 \\ -36.94 & -25.52 & -0.293 & -123.5 \end{bmatrix}.$$

The feedforward control law are calculated on-line using (6.7) and updated at each node point on the reference trajectory The simulation was performed by integrating the non-linear equations of motion of the manipulator (6.3) using the fourth order Runge-Kutta algorithm. The reference and the actual joint angle trajectories are presented in Figs. 3 and 4. The small joint angle errors over the trajectory shown in Fig. 5 indicate the efficacy of the proposed control scheme.

6.5 Conclusion

In this chapter, we have applied the results of SQ stabilization problem to design of a feedback controller for an aircraft model and a Robot model. In the first case, we consider an uncertain aircraft model to design a single state feedback controller. In the second case we consider the tracking problem of a robot manipulator. In the latter case, the feedback design problem is thought as a design of a feedback controller to stabilize a time varying system. The key feature of the proposed approach is that it obviates feedback gain scheduling. This feature is advantageous from both the performance and the implementation points of view.

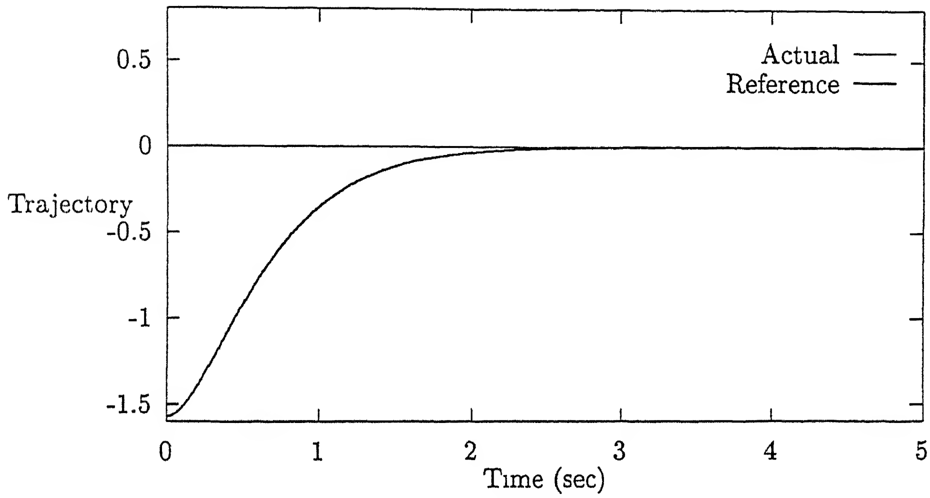

Figure 3: Response of first joint

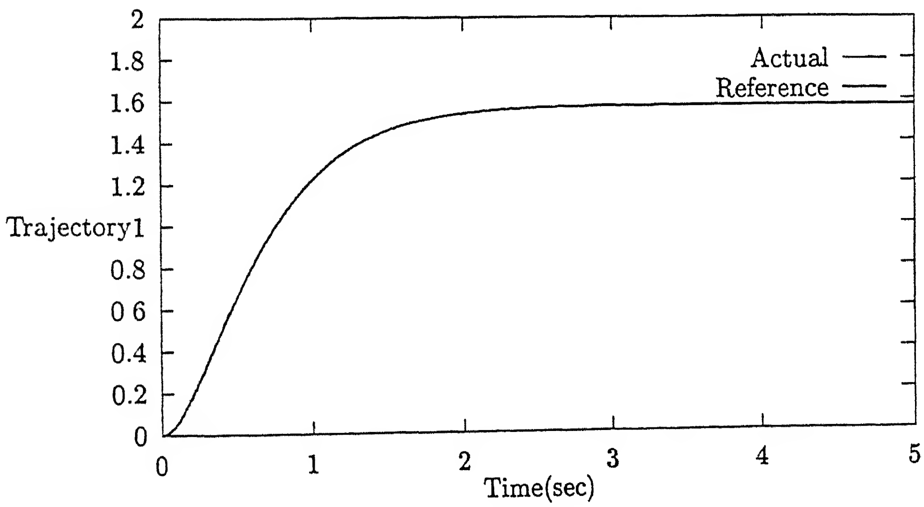

Figure 4: Response of second joint

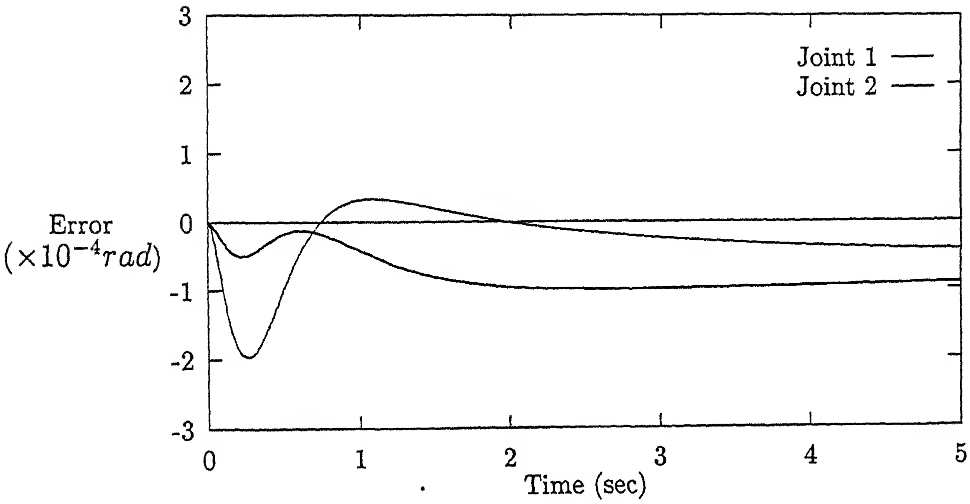

Figure 5: Joint angle errors

Chapter 7

Conclusions

7.1 Conclusions

An important problem in control theory is to stabilize finite number of linear time invariant systems by a single controller. To fulfill this requirement, this thesis considers the stabilization of a finite set of systems by a constant feedback controller. The controller is also ensured a common quadratic Lyapunov function for all the systems. The stabilization problem of a set of LTI systems is formulated as a joint search problem for a single feedback gain and a quadratic Lyapunov function and referred as simultaneous quadratic stabilization problem. A number of parametric conditions describing the set of linear systems in state space are determined. These conditions guarantee the existence of constant feedback controller for which simultaneous quadratic stabilization problem is solvable. For this purpose, we consider a set of LTI systems $\{A_i, B_i\}$ described by state space equation where
$A_i = \begin{bmatrix} A_{11}^i & A_{12}^i \\ A_{21}^i & A_{22}^i \end{bmatrix}$, $B_i = BG_i$, $B = \begin{bmatrix} 0 \\ I_m \end{bmatrix}$ and $(G_i + G_i^*) > 0$. The classes of systems which are simultaneously quadratically stabilizable by static state feedback controller are as follows :

- Matrices A_{11}^i and A_{12}^i are same for each system.
- Each system matrix A_i are in Hessenberg form.
- Matrices $\{A_{11}^i\}$ are Hurwitz and commutative pairwise.
- Matrices $\{A_{11}^i\}$ are Hurwitz and normal.

- Matrices $\{A_{12}^i\}$ are quasi sign invariant. In this case the number of inputs m are constraints as follows :
  - $m \geq n/2$ if n is even and
  - $m \geq (n+1)/2$ if n is odd.

The above classes of systems (except first one) are different from the matched uncertain systems. It has also been shown that if a set of system is stabilizable by a state feedback then their positive combination will also be stabilizable.

To design static output feedback controller, we consider a set of square (equal number of inputs and outputs) minimum phase systems $\{A_i, B_i, C\}$. Under the assumptions that the high frequency gain is non zero, the following conditions are derived for the SQ stabilization of $\{A_i, B_i, C\}$ by static output feedback controller :

- The matrices A_{11}^i and A_{12}^i are same for each system.
- The matrices $\{A_{11}^i - A_{12}^i C_2^{-1} C_1\}$ are Hurwitz and pairwise commutative.
- The matrix $\{A_{11}^i - A_{12}^i C_2^{-1} C_1\}$ are Hurwitz and normal.

The SQ stabilization problem is also extended for special class of time delay systems $\{A_i, B, A_h\}$. The existence of a stabilizing controller is assured for this class of systems. The SQ stabilization problem is also shown to be useful for the stabilization of time varying systems. As an application, we design controllers for an aircraft and a robot manipulator.

In this thesis, different class of systems in state space framework are identified for which SQ stabilization problem is solvable. These results are in fact applicable to rather specialized situations. However, it is true that if a finite set of systems is SQ stabilizable then the set of systems, consisting the neighbourhood of each system, is also SQ stabilizable. In other words, if (A_1, B_1) and (A_2, B_2) are SQ stabilizable then there exists neighbourhood of each plant N_1 and N_2 such that the systems belonging to $N_1 \cup N_2$ are also SQ stabilizable. Based on this observation, the results of this thesis will be useful for the robustness analysis of general class of systems.

7.2 Scope of Further Research

In this thesis, we characterize different classes of systems for which SQ stabilization problem is solvable Once the existence conditions are satisfied, the controller gain is computed from the algorithm. However, it is not ascertained that whether controller gain K is of minimum norm. So a procedure has to be found out to compute SQ stabilizing controller gain of minimum norm. This will be useful to avoid the saturation of sensors.

In practice controller should ensure certain other performance criteria in addition to stability So it will be interesting to design a SQ stabilizing controller which will ensure certain performance robustness.

In recent times, the ideas of switching between different controllers are used in intelligent control and applied to complex systems in the presence of structural failures [48, 46, 47]. In this approach, it is assumed that the system can be in only a finite number of configurations after failure. Each configuration corresponds to a compact set of points in the space of system parameters. One way of reducing the number of switching is to design a single controller for each configuration. Thus after determining a configuration to which the plant belongs (using pattern recognition methods), it is necessary to design a single controller for each configuration. At this stage, SQ stabilization will be useful to determine controllers as well as analyze the stability of such switching systems.

Appendix A

Long Proof

A.1 Proof of Lemma 2.3

Proof: From the results of controllability [79], if (A_i, B) is controllable then for any matrix H, $(A_i + BH, B)$ is also controllable. Using the partitioned matrices of (A_i, B) given in (2.12), the following simplifications are made.

$$\left(\begin{bmatrix} A_{11}^i & A_{12}^i \\ A_{21}^i & A_{22}^i \end{bmatrix}, \begin{bmatrix} 0 \\ I_m \end{bmatrix}\right) \text{ is controllable.}$$

$$\Rightarrow \left(\begin{bmatrix} A_{11}^i & A_{12}^i \\ A_{21}^i & A_{22}^i \end{bmatrix} - \begin{bmatrix} 0 \\ I_m \end{bmatrix} [A_{21}^i \; A_{22}^i], \begin{bmatrix} 0 \\ I_m \end{bmatrix}\right) \text{ is controllable.}$$

$$\Rightarrow \left(\begin{bmatrix} A_{11}^i & A_{12}^i \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ I_m \end{bmatrix}\right) \text{ is controllable.}$$

$$\Rightarrow \text{Rank} \begin{bmatrix} 0 & A_{12}^i & A_{11}^i A_{12}^i & (A_{11}^i)^2 A_{12}^i & \cdots \\ I_m & 0 & 0 & 0 & \cdots \end{bmatrix} = n.$$

$$\Rightarrow \text{Rank} \begin{bmatrix} A_{12}^i & A_{11}^i A_{12}^i & (A_{11}^i)^2 A_{12}^i & \cdots \end{bmatrix} = n - m.$$

$\Rightarrow (A_{11}^i, A_{12}^i)$ is controllable.

Similarly, we can show that if a system (A_i, B) is stabilizable then the system (A_{11}^i, A_{12}^i) is also stabilizable. This can be proved using the eigenvalue eigenvector criteria to test the stabilizability [2, 34] and is discussed below.

The stabilizability of the system (A_i, B) is equivalent to the following [2, 34] :
$x^* [\lambda I_n - A_i \; B] = 0$ for some $x^* \neq 0$ implies $Re\lambda < 0$.
Since (A_i, B) is stabilizable, so
$x^* [\lambda I_n - A_i] = 0$ and $x^* B = 0$ for some $x^* \neq 0$ implies $Re\ \lambda < 0$.
Note that here 0 is a vector or matrix of appropriate dimension. The vector x which satisfies $x^*B = 0$ is given by the form
$x = \begin{bmatrix} z \\ 0 \end{bmatrix}$ where $z \in R^{n-m}$. Using this x, the following simplification is made

$$\begin{aligned} x^* [\lambda I_n - A_i] &= 0 \\ \Rightarrow z^* \left[\lambda I_{n-m} - A_{11}^i \;\; A_{12}^i\right] &= 0 \end{aligned}$$

Since (A_i, B) is stabilizable, so $Re\ \lambda < 0$. Hence it is evident from the above that (A_{11}^i, A_{12}^i) is also stabilizable. □

A.2 Proof of Theorem 2.5

Sufficient: Suppose there exists a matrix $W_1 = W_1^* > 0$ and a matrix F such that the following inequalities holds,

$$[(A_{11}^i + A_{12}^i F)W_1 + W_1(A_{11}^i + A_{12}^i F)^*] < 0 \ \forall\, i.$$

We need to find a positive definite matrix W and a scalar $\gamma > 0$ such that $\Phi_i, i = 1, 2, \ldots, l$ will be negative definite where Φ_i is defined by

$$\Phi_i = [A_i W + W A_i^* - (\gamma/2)B(G_i + G_i^*)B^*]. \tag{A1.1}$$

For this purpose we construct a positive definite matrix W in the following way. By using the matrices F and W_1, choose $W_2^* = FW_1$. Then one can select W_3 such that $W_3 - W_2^* W_1^{-1} W_2 > 0$ which ensures that $W = \begin{bmatrix} W_1 & W_2 \\ W_2^* & W_3 \end{bmatrix}$ is positive definite matrix.
Such a choice of W_3 is always possible.
With the above matrices W and F, the matrix Φ_i is partitioned as

$$\Phi_i = \begin{bmatrix} \phi_{11}^i & \phi_{12}^i \\ (\phi_{12})^* & \phi_{22}^i \end{bmatrix} \text{ with } \phi_{11}^i \in R^{n-mxn-m}.$$

The different submatrices are defined as follows (by noting the special structure of matrix B defined in assumption 2.3, page 32)

$$\phi_{11}^{i} = (A_{11}^{i} + A_{12}^{i}F)W_1 + W_1(A_{11}^{i} + A_{12}^{i}F)^* \tag{A1.2}$$

$$\phi_{12}^{i} = A_{11}^{i}W_2 + A_{12}^{i}W_3 + W_1(A_{21}^{i})^* + W_2(A_{22}^{i})^* \tag{A1.3}$$

$$\phi_{22}^{i} = A_{21}^{i}W_2 + A_{22}^{i}W_3 + W_2^*(A_{21}^{i})^* + W_3(A_{22}^{i})^* - \gamma(G_i + G_i^*) \tag{A1.4}$$

In the above equation $F = W_2^* W_1^{-1}$.

The matrix Φ_i will be negative definite if [38]

$$(I)\ \phi_{11}^{i} < 0 \text{ and } (II)\ \phi_{22}^{i} - (\phi_{12}^{i})^*\ (\phi_{11}^{i})^{-1}\ \phi_{12}^{i} < 0.$$

It is observed that the matrix ϕ_{11}^{i} is negative definite for all i by the hypotheses of theorem. Also it should be noted that ϕ_{11}^{i} and ϕ_{12}^{i} are independent of γ and the matrix $(G_i + G_i^*) > 0$. So the matrix ϕ_{22}^{i} can be made negative definite by choosing a sufficiently large value of γ. Since ϕ_{11}^{i} and ϕ_{12}^{i} are independent of γ, we can choose γ such that

$$\phi_{22}^{i} - (\phi_{12}^{i})^*\ (\phi_{11}^{i})^{-1}\ \phi_{12}^{i} < 0\ \forall\ i. \tag{A1.5}$$

The value of γ is computed as follows. Using the expression of ϕ_{22}^{i} from (A1.4), the above inequalities (A1.5) are simplified as

$$\gamma I_m(G_i + G_i^*) > S_i - (\phi_{12}^{i})^*\ (\phi_{11}^{i})^{-1}\ \phi_{12}^{i}\ \ \forall\ i \tag{A1.6}$$

where $S_i = A_{21}^{i}W_2 + A_{22}^{i}W_3 + W_2^*(A_{21}^{i})^* + W_3(A_{22}^{i})^*$.

Now, for each index i, choose a scalar γ_i such that

$$\gamma_i > \lambda_M[\ (S_i - (\phi_{12}^{i})^*\ (\phi_{11}^{i})^{-1}\ \phi_{12}^{i})(G_i + G_i^*)^{-1}\ ]$$

where $\lambda_M[.]$ is the maximum eigenvalue of matrix [.].

Let the value of $\gamma = \max\{\gamma_i\}$, $i = 1, 2, \ldots, l$. For this value of γ and W, the matrix Φ_i, defined in (A1.1), will be negative definite for each $i = 1, 2, \ldots, l$. The controller gain then will be $K = (\gamma/2)B^*W^{-1}$.

Necessary : Suppose, there exists a scalar $\gamma > 0$ and a matrix $W > \mathbf{0}$ such that the following matrix inequalities hold :

$$\Phi_i = [A_i W + W A_i^* - (\gamma/2) B (G_i + G_i^*) B^*] < \mathbf{0} \ \forall\, i. \tag{A1.7}$$

Substituting $B = \begin{bmatrix} \mathbf{0} \\ I_m \end{bmatrix}$ and partitioning $W = \begin{bmatrix} W_1 & W_2 \\ W_2^* & W_3 \end{bmatrix}$, the matrix Φ_i are partitioned as

$\Phi_i = \begin{bmatrix} \phi_{11}^i & \phi_{12}^i \\ (\phi_{12}^i)^* & \phi_{22}^i \end{bmatrix}$. By using $F = W_2^* W_1^{-1}$, the different submatrices are defined as in (A1 2-A1.4). Since the matrix Φ_i is negative definite for each i, so matrix ϕ_{11}^i is also negative definite for each i. Hence the proof. □.

Appendix B

Long Proof

B.1 Proof of Theorem 5.1

Suppose there exists a matrix $W_1 = W_1^* > 0$ and a matrix F such that the following inequalities hold,

$$[(A_{11}^i + A_{12}^i F)W_1 + W_1(A_{11}^i + A_{12}^i F)^* + W_1(I_{n-m} + F^*F)W_1] < 0 \; \forall \, i$$

We need to find a positive definite matrix W and a scalar $\gamma > 0$ such that $\Phi_i, i = 1, 2, \ldots, l$ will be negative definite where Φ_i is defined by

$$\Phi_i = [A_i W + W A_i^* - \gamma B B^* + A_h A_h^* + WW]. \tag{B2.1}$$

For this purpose we construct a positive definite matrix W in the following way. By using the matrix F and W_1, choose $W_2^* = FW_1$. Then one can select W_3 such that $W_3 - W_2^* W_1^{-1} W_2 > 0$ which ensures that $W = \begin{bmatrix} W_1 & W_2 \\ W_2^* & W_3 \end{bmatrix}$ is positive definite matrix. Such a choice of W_3 is always possible.

With the above matrix W and F, the matrix Φ_i is partitioned as

$$\Phi_i = \begin{bmatrix} \phi_{11}^i & \phi_{12}^i \\ (\phi_{12})^* & \phi_{22}^i \end{bmatrix} \text{ with } \phi_{11}^i \in R^{n-m \times n-m}.$$

The different submatrices are defined as follows by taking care of special structure of B and A_h matrices given (5.9) (page 64)

$$\phi_{11}^i = (A_{11}^i + A_{12}^i F)W_1 + W_1(A_{11}^i + A_{12}^i F)^* + W_1(I_{n-m} + F^*F)W_1 \tag{B2.2}$$

$$\phi_{12}^i = A_{11}^i W_2 + A_{12}^i W_3 + W_1(A_{21}^i)^* + W_2(A_{22}^i)^* + [W_1\ W_2][W_2^*\ W_3]^* \tag{B2.3}$$

$$\phi_{22}^i = A_{21}^i W_2 + A_{22}^i W_3 + W_2^*(A_{21}^i)^* + W_3(A_{22}^i)^* - \gamma I_m + DD^* + [W_2^*\ W_3][W_2^*\ W_3]^*. \tag{B2.4}$$

The matrix Φ_i will be negative definite if [38]

$$(I)\ \phi_{11}^i < 0 \text{ and } (II)\ \phi_{22}^i - (\phi_{12}^i)^*\ (\phi_{11}^i)^{-1}\ \phi_{12}^i < 0.$$

It is observed that the matrix ϕ_{11}^i is negative definite for all i by the hypotheses of theorem. Also it is noted that ϕ_{11}^i and ϕ_{12}^i are independent of γ and the matrix $(G_i + G_i^*) > 0$. So the matrix ϕ_{22}^i can be made negative definite by choosing a sufficiently large value of γ. Since ϕ_{11}^i and ϕ_{12}^i are independent of γ, we can choose γ such that

$$\phi_{22}^i - (\phi_{12}^i)^*\ (\phi_{11}^i)^{-1}\ \phi_{12}^i < 0\ \forall\ i. \tag{B2.5}$$

The value of γ is computed as follows. Using the expression of ϕ_{22}^i from (B2.4), the above inequalities (B2.5) are simplified as

$$\gamma > S_i - (\phi_{12}^i)^*\ (\phi_{11}^i)^{-1}\ \phi_{12}^i\ \forall\ i \tag{B2.6}$$

where $S_i = (A_{21}^i W_2 + A_{22}^i W_3 + W_2^*(A_{21}^i)^* + W_3(A_{22}^i)^* + DD^* + [W_2^*\ W_3][W_2^*\ W_3]^*$.

For each index i, choose a scalar γ_i such that

$$\gamma_i > \lambda_M(\ S_i - (\phi_{12}^i)^*\ (\phi_{11}^i)^{-1}\ \phi_{12}^i\ )$$

Let the value of $\gamma = \max\ \{\gamma_i\}, i = 1, 2, \ldots, l$. For this value of γ and W, the matrix Φ_i, defined in $(B2.1)$, will be negative definite for all i. The controller gain then will be $K = (\gamma/2)B^*W^{-1}$. □

Appendix C

Quadratic Form of a Matrix

In this appendix, a few results regarding the quadratic form of a matrix are presented. However, details are available in standard matrix theory book (for example [30]).

An $n \times n$ matrix M is said to be positive definite matrix iff the following quadratic form

$$x^* M x > 0$$

for all non zero vector x.

An $n \times n$ matrix M is said to be negative definite matrix if $-M$ is a positive definite matrix.

Next the following facts are true for a positive definite matrix.

Fact 1 : A symmetric matrix A is positive definite iff the following are true

(a) All the eigenvalues of matrix M are positive.

(b) All leading principal minors are positive.

Fact 2 : Suppose that a symmetric matrix M is partitioned as $M = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ where A and C are square matrices. The matrix M is positive definite iff

(i) $A > 0$ (ii) $C - B^* A^{-1} B > 0$.

Fact 3 : Suppose A and B are symmetric matrices. Then the matrix $A+B$ will be positive definite if

$$\lambda_m[A] + \lambda_M[B] > 0.$$

Fact 4 : Suppose A and B are two symmetric and A is positive definite along with B positive semidefinite, then $(A + B)$ is also positive definite.

Bibliography

[1] Ackermann, J., Parameter space design of robust control systems, *IEEE Trans. Automat. Control*, v. 25, pp. 1058-1072, 1980.

[2] Anderson B.D.O. and J.B. Moore, *Optimal Control, Linear Quadratic Methods*, Prentice-Hall, New Delhi, 1991.

[3] Astrom K.J. and B. Wittenmark, *Adaptive Control*, Addison-Wiley, 1989.

[4] Barmish, B. R, Necessary and sufficient conditions for quadratic stabilizability of an uncertain system, *Journal of Optimization Theory and Applications*, v. 46, pp. 399-408, 1985.

[5] Barmish, B. R, A generalization of Kharitonov's four polynomial concept for robust stability problem with linearly dependent coefficient perturbations, *IEEE Trans. on Automat. Control*, v. 34, pp. 157-165, 1989.

[6] Bernussou, J., Peres, P. L. D., and Geromel, J. C., A linear programming oriented procedure for quadratic stabilization of uncertain systems, *Systems and Control Letters*, v. 13, pp. 65-72, 1989.

[7] Bhattacharyya S.P., *Robust Stabilization against Structured Perturbations*, Lecture Notes in Control and Information sciences, Springer-verleg, v. 99, 1987.

[8] Bose N. K., A system-theoretic approach to stability of sets of polynomials, *Contempory Mathematics*, v. 47, pp. 25-34, 1985.

[9] Bose N. K., and E. J.F. Delansky, Boundary implications for interval positive real functions, *IEEE Trans. on Automat. Control*, v. 36, pp. 454-458, 1989.

[10] Brierley S.D., Chiasson and S.H. Zak, On the stability ındependent of delay for linear systems, *IEEE Trans. on Automat. Control*, v. 27, pp. 252-254, 1982.

[11] Cheres E., Gutman S. and Z J. Palmor, Stabilization of uncertain dynamic system including state delay, *IEEE Trans. on Automat. Control*, v. 34, pp. 1199-1263, 1989.

[12] Chow J. H., A pole placement design approach for systems with multiple operating conditions, *IEEE Trans. on Automat. Control*, v. 35, pp. 278-288, 1990.

[13] Cook P. A., *Nonlınear Dynamıcal Systems*, Prentice-Hall International, 1986.

[14] Desoer C.A. and M. Vidyasagar, *Feedback systems: Input-output propertıes*, Academic press, 1975.

[15] Dorato P., A historical review of robust control, *IEEE Control system magazıne*, pp.44-47, 1987.

[16] Feliachi A. and A. Thowsen, Memorylees stabilization of linear delay-differential systems, *IEEE Trans. on Automat. Control*, v. 26, pp. 586-587, 1981.

[17] Feron E., Balakrishnan V. and S. Boyd, Design of stabilizing state feedback for delay systems via convex optimization, Proceeding of CDC, Tuscon, 1992.

[18] Francis B., *A course in H_∞ control theory*, Lectures notes in control and information sciences, Springer-Verlag, 88, 1987.

[19] Fu K. S., Gonzalez R. C. and C. S. G. Lee, *Robotics: Control, sensing, vision and intelligence*. McGraw-Hill Int. Eds., 1988

[20] Galimidi A. R. and B. R. Barmish, The constraint Lyapunov problem and its application to robust output feedback stabilization, *IEEE Trans. on Automat. Control*, v. 31, pp. 410-419, 1986.

[21] Geromel, J. C., Peres P. L. D., and J. Bernussou, On convex parameter space method for linear control design of uncertain systems, *SIAM Journal of Control and Optimization*, v. 29, pp. 381-402, 1991.

[22] Ghantmacher F.R., *The Theory of Matrıces*, Vols. I and II, Chelsea, New York,1959.

[23] Ghosh B. K. and C. I. Byrnes, Simultaneous stabilization and pole placement by nonswitching dynamic compensation, *IEEE Trans. on Automat. Control*, v. 28, pp. 735-741, 1983.

[24] Ghosh B. K., An approach to simultaneous system design I: semi-algebraic geometric method, *SIAM J. of Control and Optimization*, v. 24, pp. 480-496, 1986.

[25] Golub G.H. and C.H. Van Loan, *Matrix Computations*, Johns Hopkins University Press, Baltimore, 1983.

[26] Gu G., On the existence of linear optimal control with output feedback, *SIAM Journal of Control and Optimization*, v. 28, pp. 711-719, 1990.

[27] Gu G., Stabilizability conditions of multivariable uncertain systems via output feedback control, *IEEE Trans. on Automat. Control*, v. 35, pp. 925-927, 1990.

[28] Hale J.K., *Functional Differential Equations, Applied mathematical sciences*, Springer-Verleg, v. 3, 1971.

[29] Hewer G. and C. Kenney, The sensitivity of the stable Lyapunov equation, *SIAM J. control and optimization*, v. 26, pp. 321-343, 1985.

[30] Horn R. A., and C. Johnson, *Matrix Analysis*, Cambridge Univ. Press, 1985.

[31] Hoffman K. and R. Kunze, *Linear Algebra*, 2nd eds., Prentice-Hall of India, New Delhi, 1992.

[32] Jabbari, F., and Schmitendorf, W. E., A non-iterative method for the design of linear robust controllers, *IEEE Trans. on Automat. Control*, v. 35, pp. 954-957, 1990.

[33] Jacobson, D. H., *Extension of linear Quadratic Control, Optimization and matrix theory*. New York, Academic, 1977.

[34] Kailath T. *Linear Systems*, Prentice Hall Ed, New York, 1980.

[35] Khargonekar P. P., Petersen I. R. and K. Zhou, Robust stabilization of uncertain linear systems: Quadratic stabilizability and H_∞ control theory, *IEEE Trans. on Automat. Control*, v. 35, pp. 356- 361, 1990.

[36] Khartinov V.L., Stabilization of uncertain linear systems, *Differential equations*, v 14, pp. 1483-1485, 1979.

[37] Kohivu A.J., *Fundamentals for control of robotic manipulators*, John Wiley and sons, inc. 1989.

[38] Kreindler E. and A. Jamenson, Conditions for nonnegativeness of partitioned matrices, *IEEE Trans. Automat. Control*, v. 22, pp. 468-470, 1972.

[39] Krawosovskii N.N and J. L. Brenner, *Stability of Motion, Applications of Lyapunov's second method to differential systems and equations with Delay*, Standford Univ. Press, Standford, 1963.

[40] Landau I.D. and B. Courtitiol, Adaptive model following systems for flight control and simulation, *Journal of Aircraft*, v. 9, pp. 668-674, 1972.

[41] Laub A.J. and A. Linnemann, Hessenberg and Hessenberg/triangular forms in linear system theory, *International Journal of Control*, pp. 1523-1547, 1986.

[42] Leitmann G., Guaranteed asymptotic stability for some linear systems with bounded uncertainties, *Journal Dynamics Systems, Measurement and Control*, v. 101, pp. 212-216, 1979.

[43] Levine W.S. and M. Athans, On the determination of the optimal constant output feedback gains for linear multivariable systems, *IEEE Trans. on Automatic Control*, v. 15, pp. 44-48, 1970.

[44] Mahmoud M.S. and N.F. A.Muthairi, Design of Robust controllers for time-delay systems, *IEEE Trans. on Automatic Control*, v. 39, pp. 995-999, 1994.

[45] Narendra K. S. and A.M. Annaswamy, *Stable Adaptive systems*, Prentice-Hall, New Jersey, 1989.

[46] Narendra K. S. and J. Balakrishnan, A Common Lyapunov function for stable LTI systems with commutating A-matrices, *IEEE Trans. on Automat. Control*, v. 39, pp. 2469-2471, 1994.

[47] Narendra K. S. and J. Balakrishnan, Adaptation and Learning using multiple models, switching, and tuning, *IEEE Control System Magazine* , pp. 37-51, 1995.

[48] Narendra K. S. and S. Mukhopadhyay, Intelligent control using Neural Network, *IEEE Control System Magazine* , pp. 11-18, 1992.

[49] Ni M. and H. Wu, A Riccati equation approach to the design of linear robust controllers, *Automatica*, v. 29, pp. 1603-1605, 1993.

[50] Okada T., Kihara M. and K. Motoyama, Sensitivity reduction by double perfect model following, *IEEE Trans. Aerospace and Electronic Systems*, v. 18, pp. 29-38,1982.

[51] Patel R.V., Toda M. and B. Sridhar, Robustness of linear quadratic state feedback design in the presence of system uncertainty, *IEEE Trans. on Automat. Control*, v. 22, pp. 1058-1072, 1977.

[52] Paul R. P. Paul, *Robot Manipulators: Mathematics, Programming, and Control*, The MIT Press, Cambridge, Massachusetts. 1981.

[53] Peres P.L.D., Geromel J.C. and J. Bernussou, Quadratic stabilizability of linear uncertain systems in convex-bounded domains, *Automatica*, v. 29, pp. 491-493, 1993.

[54] Petersen, I.R. and Hollot, C .V., A Riccati equation approach to the stabilization of uncertain linear systems, *Automatica*, v. 22, pp. 397-411, 1986.

[55] Phoojaruenchanachai S. and K. Furuta, Memoryless stabilization of uncertain linear systems including time varying state delays, *IEEE Trans. on Automat. Control*, v. 37, pp. 1022-1026,1992.

[56] Rosenbrock H.H. *State-space and Multivariable Theory*, Wiley, New York, 1970.

[57] Saeks R. and J. Murray, Fractional representation, algebraic geometry and the simultaneous stabilization problem, *IEEE Trans. on Automat. Control*, v. 27, pp. 895-903, 1982.

[58] Saberi A. and P. Sanuti, Squaring down by static and dynamic compensators, *IEEE Trans. on Automat. Control*, v. 33, pp. 358-365, 1988.

[59] Schmitendorf W. E., and C. V. Hollot, Simultaneous stabilization via linear state feedback control, *IEEE Trans. on Automat. Control*, v. 34, pp. 1001-1005. 1989.

[60] Seraji H., An approach to multivariable control of manipulators, *ASME Journal of Dynamic Systems, Measurement, and Control*, v. 109, 146-154, 1987.

[61] Shen J.C., Chen B.S., and F.C. Kung, Memoryless stabilization of uncertain dynamic systems: Riccati equation approach, *IEEE Trans. on Automat. Control*, v. 36, pp. 638-640, 1991.

[62] Shieh L.S., Tsay Y.T., Block modal matrices and their applications to multivariable control systems, *IEE Proc. Control theory and applications*, Pt. D, v. 129, pp. 41-48, 1982.

[63] Spong M.W. and M. Vidyasagar : *Robot Dynamics and Control*, John Wiley, 1989.

[64] Steinberg A. and M. Coreless, Output feedback stabilization of uncertain dynamical systems, *IEEE Trans. on Automat. Control*, v. 35, pp. 1025-1027, 1985.

[65] Steinberg A., A sufficient condition for output feedback stabilization of uncertain dynamical systems. *IEEE Trans. on Automat. Control*, v. 33, pp. 676-677, 1988.

[66] Swarup A. and M. Gopal, Robust trajectory control of a robot manipulator, *International Journal of System Science*, v. 22, no. 11, 2185-2194, 1991.

[67] Syrmos V.L., On the finite transmission zero assignment problem, Automatica, v. 29, pp. 1121-1126, 1993.

[68] Thorp J.S. and B.R. Barmish, On guaranteed stability of uncertain linear systems via linear control, *Journal of Optimization Theory and Applications*, v. 35, pp. 559-579, 1981.

[69] Tseng C.L., Fong I.K., and J.H. Su, Robust stability analysis for uncertain delay systems with output feedback controller, *Systems and Control Letters*, v. 23, pp. 271-278, 1994.

[70] K. S. Tsakalis and P.A. Ioannau, *Linear time varying systems, Control and Adaptation*. Printice-Hall, 1993.

[71] Vidyasagar. M. and N. Viswanadham, Algebraic design techniques for reliable stabilization, *IEEE Trans. on Automat. Control*, v. 27, pp. 1085-1095, 1982.

[72] Vidyasagar, M., *Control System Design: a Factorization Approach*, Prentice Hall, 1985.

[73] Vidyasagar, M., *Nonlinear systems*, Prientice Hall, 2nd eds, 1993.

[74] Wei K. and B. R. Barmish, An iterative design procedure for simultaneous stabilization of MIMO systems, *Automatica*, v. 24, pp. 643-652, 1988.

[75] Wei K., Stabilization of linear time invariant interval systems via constant state feedback control, *IEEE Trans. on Automat. Control*, v. 39, pp. 22-32, 1994.

[76] Wei K. and B. Barmish, Making a polynomial Hurwitz-invariant by choice of feedback gains, *International Journal of Control*, v. 50, pp. 1025-1038, 1989.

[77] Wei K., Quadratic stabilizability of linear systems with structural independent time-varying uncertainties, *IEEE Trans. on Automat. Control*, v. 35, pp. 268-277, 1990.

[78] Wei K. and R.K. Yedavalli, Invariance of strict Hurwitz property for uncertain polynomial with dependent coefficients, *IEEE Trans. on Automatic Control*, v. 32, pp. 907-909, 1987.

[79] Wonham, W. M., *Linear Multivariable Control, A Geometrical Approach*. Springer-verleg, 1974.

[80] Yedavalli R. K., Perturbation bounds for robust stability in linear state space models, *International Journal of Control*, v. 42, pp. 1507-1517, 1985.

[81] Youla D.C., Bongiorno J.J. and C.N.Lu, Single loop feedback stabilization of linear multivariable plants, *Automatica*, v. 10, pp. 159-173, 1974.

[82] Zavarei M.M. and M. Jamshedi, *Time-Delay Systems, Analysis, Optimization and Applications*, North-Holland Systems and Control series, v. 9, 1986.

[83] Zeheb E., A sufficient condition for output feedback stabilization of uncertain systems. *IEEE Trans. on Automat. Control*, v. 31, pp. 1055-1057, 1986.

AUTHOR'S PUBLICATIONS

[1] **I. N. Kar** : A Controller design algorithm for finite number of linear systems, Accepted for publication in *International Journal of System Science*, 1996.

[2] **I. N. Kar** and A. Dasgupta : Simultaneous Stabilization of multiple plants : An approach to Robot Control. Will appear in the *Proceedings of International Symposium on Robotics and Intelligent Systems*, ISIRS'95, Bangalore, INDIA, Nov. 1995.

[3] **I. N. Kar** : On the pole assignment robustness of linear systems, *Proceeding of Eighteenth National System Conference* - NSC-94, Agra, INDIA, PP- 664-666, 1995.

[4] **I. N. Kar** : A Controller design method for finite number of linear systems, *Proceeding of 19 th National System Conference*, Coimbatore, INDIA, 1995.

[5] A. Ghosh and **I. N. Kar** : Robust controller and observer design for a two terminal hvdc systems, *Proceedings of IEEE Asia pacific Region 10 Conference*, TENCON'91, New Delhi, 1991. Vol.-1, PP- 44-48.

[6] A. Ghosh, **I. N. Kar** and R. K. Pandey : A Comparative study of two discrete time hvdc converter models, *Proceedings of IEEE Asia Pacific Region 10 Conference*, TENCON'91, New Delhi, 1991. Vol.-1, PP- 154-158.

[7] **I. N. Kar** and A. Ghosh : Robust state feedback controllers for two terminal hvdc systems, *Australasian Universities Power Engineering Conference*, at University of Wollongong, AUPEC' 93, 1993.

[8] **I. N. Kar** : Stabilization of a collection of linear systems using a static output feedback controller, submitted for publication.

[9] **I. N. Kar** : Memoryless output feedback stabilization of a time-delay systems, submitted for publication.

ERRATA

The following two references will be added at the end of bibliography at the page 83.

[84] Boyd S.P., El Ghaoui L., Feron E. and V. Balakrishnan: Linear Matrix Inequalities Systems and Control Theory, *SIAM studies in applied mathematics,* 1994.

[85] Skelton and T. Iwasaki : Increased roles of Linear Algebra in Control Education, *IEl Control Systems,* 15(4), pp. 76-90, 1995.

These references will be read in page 6 after the line number 7 as follows:
Other numerical methods for solving linear matrix inequalities can be found in [84, 85 aı references therein].
The following lines are modified in the following ways.

In page 4, line 2 from top: .... control system design [29].
In page 4, line 15 from top: .... reported in [53,80,42,4,35,54,68,51].
In page 7, line 11 from top: .... related results [36,9,8,12,15,74,78]....
In page 48, line 3 from bottom: ....algorithmic point of view [43,20]....
In page 71, line 9 from top: .... 4th order model [40,50]....
In page 87, line 4 from top: .... of controllability [79,14,22,25,45,13,3], if...